University of Swaziland

Final Examination, December 2012

BSc IV, Bass IV, BEd IV, BEng III

Title of Paper : Partial Differential Equations

Course Number : M415

Time Allowed : Three (3) Hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions. **Submit solutions to ONLY FIVE questions.**
4. Show all your working.
5. A Table of Laplace Transforms is provided at the end of the question paper.

Question 1

(a) By eliminating the arbitrary function, find the partial differential equation satisfied by the following functions

(i) $u(x,y) = f\left(\frac{xy}{u}\right)$ [4]

(ii) $u = f(ax+by) + g(cx+dy)$
where $a,\ b,\ c$ and d are nonzero constants [6]

(b) Find the general solution of the partial differential

$$(au-by)u_x + (bx-cu)u_y = cy - ax$$

where $a,\ b,\ c$ and d are nonzero constants. [10]

Question 2

(a) Find the particular solution of the partial differential equation

$$x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u$$

that passes through the curve

$$\Gamma:\quad u = 0 \text{ on } x+y=0$$

[10]

(b) Given the function $u = u(x,y)$ and transformation $x = r\cos\theta,\ y = r\sin\theta$ write

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

in terms of r and θ [10]

Question 3

Reduce the following equation

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0,\quad x \neq 0,\ \ y \neq 0$$

to canonical form and then find the general solution [20]

Question 4

Consider the function

$$f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0; \\ \pi - x, & 0 \le x \le \pi. \end{cases}, \qquad f(x + 2\pi) = f(x)$$

(a) Find the fourier series expansion for $f(x)$. [10]

(b) Use Parseval's identity to find the value of the sum

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots.$$

[10]

Question 5

Use Green's theorem

$$\iint_{\Omega} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint_{\Gamma} Mdx + Ndy$$

to show that the solution for the following partial differential equation

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0, \qquad -\infty < x < +\infty, \quad t > 0 \\ &u(x,0) = f(x) \\ &u_t(x,0) = g(x) \end{aligned}$$

is given by

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)ds.$$

(Ω is the characteristic triangle and Γ is the boundary of the characteristic triangle). [20]

Question 6

Solve the following boundary value problem

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H \\ &u(x,0) = 0, \quad 0 \le x \le L \\ &u(x,H) = 0, \quad 0 \le x \le L \\ &u(L,y) = 0, \quad 0 \le y \le H \\ &u(0,y) = f(y), \quad 0 \le y \le H \end{aligned}$$

where $f(y)$ is a defined function. [20]

Question 7

Solve the following partial differential equation using the method of Laplace transforms

$$
\begin{aligned}
&u_{tt} = c^2 u_{xx} + \sin(\pi x), \quad 0 < x < 1, \quad t > 0 \\
&u(x,0) = 0, \quad 0 \le x \le 1 \\
&u_t(x,0) = 0, \quad 0 \le x \le 1 \\
&u(0,t) = 0, \quad t \ge 0 \\
&u(1,t) = 0, \quad t \ge 0.
\end{aligned}
$$

[20]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{d^n f}{dt^n}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$