University of Swaziland

Final Examination, 2012/2013

BSc IV, Bass IV, BEd IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1

Define $\oplus$ and $\odot$ on $\mathbb{Z}$ as follows. For $m, n \in \mathbb{Z}$,

$$m \oplus n = m + n - 1$$

and

$$m \odot b = m + n - mn$$

where the operations on the right hand side are the usual addition and multiplication.

1.1 Show that $\langle \mathbb{Z}, \oplus, \odot \rangle$ is a commutative ring. [12]

1.2 Show that $\langle \mathbb{Z}, \oplus, \odot \rangle$ has identity by determining this identity. [3]

1.3 Consider the map $\phi : \langle \mathbb{Z}, \oplus, \odot \rangle \to \mathbb{Z}$ defined by $\phi(n) = 1 - n$. Show that ϕ is a ring homomorphism. [5]

QUESTION 2

2.1 Let F be a field. Explain what is meant by saying, *"The polynomial $f(x)$ is irreducible in $F[x]$."* [2]

2.2 State Eisenstein's irreducibility criterion. [2]

2.3 Determine the irreducibility or otherwise of

2.3.1 $x^3 - 7x^2 + 3x + 3$ in $\mathbb{Q}[x]$. [6]

2.3.2 $2x^{10} - 25x^3 + 10x^2 - 30$ in $\mathbb{Q}[x]$. [4]

2.4 Suppose

$$f(x) = x^5 + 5x^4 + 3x + 2 \quad \text{and} \quad g(x) = 2x^2 + 1$$

are polynomials in $\mathbb{Z}_7[x]$. Find $q(x)$ and $r(x)$ in $\mathbb{Z}_7[x]$ as described by the division algorithm so that $f(x) = q(x)g(x) + r(x)$ with $r(x) = 0$ or $\deg r(x) < \deg g(x)$. [6]

QUESTION 3

3.1 Let α be an element of an extension field E of F. Explain what it means for α to be

3.1.1 *algebraic over F?* [2]

3.1.2 *transcendental over F?* [1]

3.2 State (do not prove) Kronecker's Theorem. [4]

3.3 Consider the polynomial $x^3 + x^2 + 1$ in $\mathbb{Z}_2[x]$.

3.3.1 Show that $x^3 + x^2 + 1$ is irreducible over $\mathbb{Z}_2$. [3]

3.3.2 Let α be a zero of $x^3 + x^2 + 1$ in an extension field of $\mathbb{Z}_2$. Show that $x^3 + x^2 + 1$ factors into three linear factors in $\mathbb{Z}_2(\alpha)[x]$ by finding this factorisation. [10]

QUESTION 4

4.1 Let R and S be rings. Define

4.1.1 an *ideal* in R, [2]

4.1.2 a *ring homomorphism* $\phi : R \to S$, [2]

4.1.3 the *kernel* of a ring homomorphism $\phi : R \to S$. [2]

4.2 Let $\phi : R \to S$ be a ring homomorphism.

4.2.1 Let 0_R and 0_S be the zeroes in R and S respectively. Prove that $\phi(0_R) = 0_S$. [4]

4.2.2 Show that for $r \in R$, $\phi(-r) = -\phi(r)$. [4]

4.2.3 Show that $\ker \phi$ is an ideal in R. [6]

QUESTION 5

5.1 Let α be a zero of $x^2 + x + 1$ in an extension field of $\mathbb{Z}_2$.

5.1.1 Write down all the elements of $\mathbb{Z}_2(\alpha)$. [4]

5.1.2 Construct the multiplication table for $\mathbb{Z}_2(\alpha)$. [Show how each product was found.] [8]

5.2 For each algebraic number $\alpha \in \mathbb{C}$, find $\operatorname{irr}(\alpha, \mathbb{Q})$ and $\deg(\alpha, \mathbb{Q})$.

5.2.1 $\sqrt{2} + i$ [4]

5.2.2 $\sqrt{2 + \sqrt{3}}$ [4]

QUESTION 6

6.1 Define (i) an *integral domain* and (ii) a *field*. Give an example of an integral domain that is not a field. [6]

6.2 Prove that a finite integral domain is a field. [8]

6.3 Prove that every field is an integral domain. [6]

QUESTION 7

7.1 Let R be the ring of matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $a, b \in \mathbb{R}$. Show that the map $\phi : R \to \mathbb{C}$ defined by

$$\phi \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a + ib$$

is a ring homomorphism. Find its kernel. [You do not need to show that R is a ring.] [4]

7.2 Let F be a field.

7.2.1 Show that the only ideals in F are $\{0\}$ and F itself. [6]

7.2.2 Let $\phi : F \to S$ be a ring homomorphism. Show that ϕ is either the zero map or ϕ is one-to-one. [10]

END OF EXAMINATION PAPER