UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2012/2013

B.Sc. / B.Ed. / B.A.S.S.IV

TITLE OF PAPER	:	METRIC SPACES
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Let $X = \mathbb{R}^2$, and for $x, y \in \mathbb{R}^2$, let d be the raspberry pickers' function. Show that d is a metric on $\mathbb{R}^2$. [13]

(b) Let $X = \mathcal{C}[-1,1]$, and let $x(t) = t$ and $y(t) = t^3$ for $t \in [-1,1]$. Find $d(x,y)$ in $\mathcal{C}[-1,1]$, where d is the

(i) uniform metric, [2]

(ii) L_1-metric, [2]

(iii) L_2-metric. [3]

QUESTION 2

(a) Let (X,d) be a metric space, and let $A, B, C \subseteq X$. Show that if $A \subseteq B$, then $d(B,C) \leq d(A,C)$. [5]

(b) Let (X,d) be a metric space. Show that the mapping ρ defined below is a metric in each case:

(i) $\rho(x,y) = \min\{d(x,y), 1\}$; [6]

(ii) $\rho(x,y) = \dfrac{d(x,y)}{1 + d(x,y)}$. [6]

(c) Show that infinitely many metrics can be defined on a set X with more than one member. [3]

QUESTION 3

(a) Prove that a subset G of a metric space X is open if and only if G is the union of all open balls contained in it (G). [10]

(b) Let X be a metric space, and let A, B be subsets of X. Prove that:

i. $A \subseteq B \implies A^\circ \subseteq B^\circ$. [2]

ii. $(A \cap B)^\circ = A^\circ \cap B^\circ$. [6]

iii. $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$. [2]

QUESTION 4

(a) Let X be a metric space, and let F be a subset of X. Prove that F is closed if and only if it's complement, F^c, is open in X. [10]

(b) Consider $\mathbb{R}^2$ with the New York metric, and let $(x^{(n)})_{n\geq 1}$, where $x^{(n)} = (x_1^{(n)}, x_2^{(n)})$, be a sequence of points in $\mathbb{R}^2$. Prove that $(x^{(n)})_{n\geq 1}$ converges to $x = (x_1, x_2) \in \mathbb{R}^2$ if and only if either

1. $x_1^{(n)} = x_1 \forall n \in \mathbb{N}$ and $x_2^{(n)} \to x_2$, or
2. $x_1^{(n)} \neq x_1$ for some $n \in \mathbb{N}$, and $x_1^{(n)} \to x_1$, $x_2^{(n)} \to x_2 = 0$. [10]

QUESTION 5

(a) Let X be a metric space and $(x_n)_{n\geq 1}$ be a sequence in X. What is meant by saying that $(x_n)_{n\geq 1}$ is *convergent*? [2]

(b) Decide whether or not the following sequences are convergent in the usual (Euclidean) metric on $\mathbb{R}^2$:

(i) $x_n = \left(\dfrac{n^3}{3n^3 - 1}, \dfrac{3}{3n+2} \sin\left(\dfrac{n\pi}{2}\right) \right);$ [4]

(ii) $x_n = \left(10^{-n}, (-1)^n \exp\left(\dfrac{1}{n}\right) \right).$ [3]

(c) (i) Suppose that $(x_n)_{n\geq 1}$ converges to x in $C[a,b]$ in the uniform metric. Explain what is meant by *pointwise convergence* of a sequence $(x_n)_{n\geq 1}$ in $C[a,b]$. Show that $(x_n)_{n\geq 1}$ converges to x pointwise. [1,3]

(ii) Let x_n in $C[0,1]$ be defined by

$$x_n(t) = \begin{cases} \frac{nt}{n-1} & \text{if } 0 \leq t \leq 1 - \frac{1}{n}, \\ n(1-t) & \text{if } 1 - \frac{1}{n} \leq t \leq 1. \end{cases}$$

Sketch the graph of $x_n(t)$ and show that $(x_n)_{n\geq 1}$ converges pointwise to the function

$$x(t) = \begin{cases} t & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t = 1. \end{cases}$$

Deduce that $(x_n)_{n\geq 1}$ is not convergent in $C[0,1]$ in the uniform metric. [1,5,1]

QUESTION 6

(a) Describe the open balls $B(a, 3)$ and the closed balls $B[a, 3]$ in $\mathbb{R}^2$ with respect to the raspberry pickers' metric, where

i. $a = (0, 0)$, and [2,1]

ii. $a = (2, 3)$. [2,1]

(b) Show that for any two points x and y of a metric space, there exists disjoint open balls such that one is centered at at x and the other one is centered at y. [5]

(c) Prove that the limit point of a set S in a metric space X is either an interior point or a boundary point of S. [3]

(d) Prove that an isolated point of a set S in a metric space X is a boundary point of S^c. [3]

(e) Prove that a boundary point of a subset S of a metric space X is either a limit point of S or an isolated point of S. [3]

QUESTION 7

(a) Let f be the function $f : C[0,1] \longrightarrow \mathbb{R}$ defined for $x \in C[0,1]$ by $f(x) = x(0)$. Show that f is not continuous with respect to the L_1−metric on $C[0,1]$ (and the usual metric on $\mathbb{R}$) by considering the functions given by

$$x_n(t) = \begin{cases} (n-1)t & \text{if } 0 \le t \le \frac{1}{n}, \\ 1-t & \text{if } \frac{1}{n} \le t \le 1. \end{cases}$$

(**Hint**: Sketch the functions $x_n(t)$ and consider their limit in the L_1−metric). [6]

(b) Let (X, d) be a metric space with the metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 3 & \text{if } x \neq y. \end{cases}$$

Show that any Cauchy sequence in X is eventually constant, and deduce that (X, d) is complete. [4]

(c) i. Explain what is meant by a *contraction* of a metric space. Show that if $f : [a,b] \longrightarrow [a,b]$ is differentiable, then f is a contraction if and only if there is a positive real number $r < 1$ such that $|f'(x)| \le r$ for every $x \in (a,b)$. [4]

ii. State without proof the *Contraction Mapping Theorem.* [2]

iii. Show that the mapping $f : [-1,1] \longrightarrow [-1,1]$ defined by

$$f(x) = \frac{1}{14}(3x^3 - 2x^2 + 9)$$

is a contraction, and deduce that there is a unique solution to the equation $3x^3 - 2x^2 - 14x + 9 = 0$ in the interval $[-1,1]$. [4]

END OF EXAMINATION