UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2012/2013

B.Sc. / B.Ed. / B.A.S.S.IV

TITLE OF PAPER	:	METRIC SPACES
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Let $X = \mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R},\ 1 \le i \le n\}$ be the set of real $n-$tuples. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}.$$

Prove that (X, d) is a metric space. [10]

(b) Let $A \subset \mathbb{R}^2$ be the region bounded by the unit disc centered at $(1, 0)$, and let $x = (2, 1)$. Find $d(x, A)$ in each of the following metrics:

(i) Chicago metric; [2]

(ii) Max (or uniform) metric; [2]

(iii) London (or UK rail) metric; [2]

(iv) New York metric; [2]

(v) Raspberry pickers' metric; [2]

QUESTION 2

Let $A \subset \mathbb{R}^2$ be the region bounded by the unit disc centered at $(1, 0)$. Find $\mathrm{diam}(A)$ in each of the following metrics:

(i) Chicago metric; [2]

(ii) Max (or uniform) metric; [2]

(iii) London (or UK rail) metric; [4]

(iv) New York metric; [6]

(v) Raspberry pickers' metric; [6]

QUESTION 3

(a) Let A be a subset of a metric space X. Prove that A° is an open subset of A that contains every open subset of A. [10]

(b) Let F be a subset of a metric space. Prove that F', which is the set of limit points of F, is a closed subset of F; that is $(F')' \subseteq F'$. [10]

QUESTION 4

(a) Let X be a metric space, and let $F \subseteq X$. Prove that if x_0 is a limit point of F, then every open ball $B(x_0, r)$, where $r > 0$, contains an infinite number of points of F. [4]

(b) Let X be a metric space, and let F_1 and F_2 be subsets of X. Prove that:

i. If $F_1 \subseteq F_2$, then $F_1' \subseteq F_2'$; [4]

ii. $(F_1 \cup F_2)' = F_1' \cup F_2'$; and [10]

iii. $(F_1 \cap F_2)' \subseteq F_1' \cap F_2'$. [2]

QUESTION 5

(a) Prove that:

i. Every convergent sequence is a Cauchy sequence, [3]

ii. If $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are Cauchy sequences in a metric space X, then the sequence $(d(x_n, y_n))_{n\geq 1}$ is convergent in $\mathbb{R}$. [5]

(b) What do you understand by the following:

i. A nowhere dense metric space; [1]

ii. An everywhere dense metric space? [1]

(c) i. Suppose that $(x_n)_{n\geq 1}$ converges to x in $C[a, b]$ in the uniform metric. Explain what is meant by *pointwise convergence* of a sequence $(x_n)_{n\geq 1}$ in $C[a, b]$. Show that $(x_n)_{n\geq 1}$ converges to x pointwise. [2,3]

ii. Let x_n in $C[0, 1]$ be defined by

$$x_n(t) = \begin{cases} \frac{nt}{n-1} & \text{if } 0 \leq t \leq 1 - \frac{1}{n}, \\ n(1-t) & \text{if } 1 - \frac{1}{n} \leq t \leq 1. \end{cases}$$

Sketch the graph of $x_n(t)$ and show that $(x_n)_{n\geq 1}$ converges pointwise to the function

$$x(t) = \begin{cases} t & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t = 1. \end{cases}$$

Deduce that $(x_n)_{n\geq 1}$ is not convergent in $C[0, 1]$ in the uniform metric. [1,3,1]

QUESTION 6

(a) Prove that in a metric space X, a subset $F \subseteq X$ is closed if and only if the limit of any convergent sequence $(x_n)_{n\geq 1}$ of points of F is in F. [8]

(b) Prove that $\mathbb{R}^2$ equipped with the metric

$$d(x,y) = \alpha|x_1 - y_1| + |x_2 - y_2|; \qquad x = (x_1, x_2), \quad y = (y_1, y_2)$$

is complete, where the real number $\alpha > 0$ is fixed. [12]

QUESTION 7

(a) Let X be a nonempty set and let d_1 and d_2 be metrics on X.

i. What is meant by saying that the metrics d_1 and d_2 are *equivalent*? [3]

ii. Suppose that there are positive integers k and K such that

$$k\, d_1(x,y) \leq d_2(x,y) \leq K\, d_1(x,y)$$

for all $x, y \in X$. Show that d_1 and d_2 are equivalent. [7]

(b) Show that on $\mathbb{R}$ the Euclidean metric and the Chicago metric are equivalent. [4]

(c) Explain what is meant by saying that a metric space is *connected*. Which of the following subspaces or $\mathbb{R}$ is connected, and which is disconnected? Give reasons. (Any theorem about connected subsets of $\mathbb{R}$ that you use should be stated carefully but not proved.)

i. $\mathbb{R} - \mathbb{Q}$, [3]

ii. $(2,4) \cup (3,\infty)$, [2]

iii. $[999, 1001)$. [1]

END OF EXAMINATION