University of Swaziland

Final Examination, April 2014

B.Sc II, B.A.S.S II, B.Ed II, B.Eng II

Title of Paper : Ordinary Differential Equations

Course Code : M213

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
    a. **SECTION A(COMPULSORY): 40 MARKS**
    Answer ALL QUESTIONS.
    b. **SECTION B: 60 MARKS**
    Answer ANY THREE questions.
    Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Non programmable calculators may be used (unless otherwise stated).
5. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

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Question 1

(a) Define each of the following terms and give an example

- (i) Initial value problem. [3]
- (ii) Homogeneous function. [3]
- (iii) Linear ordinary differential equation. [3]
- (iv) Ordinary point for a second order linear ordinary differential equation. [3]

(b) Given that if $y_1(x)$ and $y_2(x)$ are two different solutions of

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Is the function $r(x) = c(y_1(x) - y_2(x)) + y_2(x)$ a solution of the differential equation where c is a constant? Explain. [3]

(c) Prove that if

$$M(x,y)dx + N(x,y)dy = 0$$

is a homogeneous differential equation, then the change of variables $y = ux$ transforms the differential into a separable differential equation with variables u and y. [7]

(d) (i) Given the function

$$g(t) = \begin{cases} 0, & 0 \leq t \leq a; \\ f(t-a), & a < t. \end{cases}$$

Show that

$$\mathcal{L}[g(t)] = e^{-as}\mathcal{L}[f(t)].$$

[5]

(ii) Using the result in d(i), find $\mathcal{L}[g(t)]$ where

$$g(t) = \begin{cases} 0, & 0 \leq t \leq \pi; \\ e^{t-\pi}, & \pi < t. \end{cases}$$

[3]

(e) Show that the solution for the differential equation

$$p(x)y'(x) + q(x)y(x) = r(x)$$

is given by

$$y(x) = e^{-\int \frac{q(x)}{p(x)}dx}\left(\int \frac{r(x)}{p(x)}e^{\int \frac{q(x)}{p(x)}dx} + c\right).$$

Hence solve

$$xy' + 2y = 4x^2$$

[10]

SECTION B: ANSWER ANY 3 QUESTIONS

Question 2

(a) Using the substitution $u = \ln x$. Find the general solution of

$$2x^2y'' - 3xy' + 2y = 0$$

[10]

(b) Solve the following differential equation

$$y'' + y = \sec x.$$

Use the method of variation of parameters to find the particular solution. [10]

Question 3

Solve the following differential equations

(a) $xy' + (x+1)y = 1.$ [5]

(b) $y'' + 2y' = x + 1.$ [5]

(c) $y' = xe^{x+y}.$ [5]

(d) $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0.$ [5]

Question 4

(a) Show that the change of variable

$$y = -\frac{1}{P(x)z}\frac{dz}{dx}$$

transforms the equation

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$$

into a second order linear differential equation

$$\frac{d^2z}{dx^2} - \left(Q(x) + \frac{1}{P(x)}\frac{dP}{dx}\right)\frac{dz}{dx} + P(x)R(x)z = 0.$$

Hence solve the equation

$$\frac{dy}{dx} = xy^2 + \left(2 - \frac{1}{x}\right)y - \frac{3}{x}.$$

[13]

(b) Solve the following differential equation 38

$$(y\cos(xy)+1)dx+(x\cos(xy))dy=0.$$

[7]

Question 5

(a) Given $y=x$ is a solution of

$$(x^2-1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0,$$

find another linearly independent solution by reducing the order. [10]

(b) Solve using the method of Laplace transforms

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=5, \quad y(0)=0, \ y'(0)=0.$$

[10]

Question 6

(a) Locate and classify the singular point(s) of the following differential equation

$$9x^2\frac{d^2y}{dx^2}+9x\frac{dy}{dx}+(9x^2-1)y=0.$$

[3]

(b) By using the appropriate method, find the series solution of the differential equation in (a) about $x=0$. [17]

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Table 1: Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}[f(t)]$
t^n	$\dfrac{n!}{s^{n+1}}$
$\dfrac{1}{\sqrt{t}}$	$\sqrt{\dfrac{\pi}{s}}$
e^{at}	$\dfrac{1}{s-a}$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$\dfrac{1}{a-b}\left(e^{at} - e^{bt}\right)$	$\dfrac{1}{(s-a)(s-b)}$
$\dfrac{1}{a-b}\left(ae^{at} - be^{bt}\right)$	$\dfrac{s}{(s-a)(s-b)}$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$\sin(at) - at\cos(at)$	$\dfrac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\dfrac{a}{s^2-a^2}$
$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\dfrac{2a^2}{s^4+4a^4}$
$\dfrac{d^n f}{dt^n}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
$g(t) = \begin{cases} 0, & 0 \le t \le a; \\ f(t-a), & a < t. \end{cases}$	$e^{-as}F(s)$