UNIVERSITY OF SWAZILAND 48

FINAL EXAMINATION, 2013/2014

B.Sc. II, BASS II, BED. II

Title of Paper : Linear Algebra

Course Number : M220

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked.**
2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS 49

A1. (a) Define a vector space (4)

(b) Define a linear transformation (3)

(c) Which of the following are linear transformations?

(i) $T : p_2(x) \to p_1(x);\quad T(ax^2 + bx + c) = 2ax + b$

(ii) $T : \mathbb{R}^3 \to \mathbb{R}^3;\quad T(x, y, z) = (x + 1, 2y, z)$ (8)

(d) Show that $(12, 12, -3)$ is linear combination of the vectors $(2, 0, 1), (4, 2, 0), (1, 3, -1)$ (5)

A2. (a) Let $s = \{v_1, v_2, \cdots, v_n\}$ be a set of vectors in a vector space V. Explain precisely what is meant by each of the following statements.

(i) s spans V

(ii) s is a basis for V

(iii) s is linearly independent. (9)

(b) Determine whether or not the set $\{(1, 2, 0), (1, 2, 1), (3, 4, 3)\}$ spans $\mathbb{R}^3$ (6)

(c) Determine whether or not the set $\{(1, -1, 1), (2, 0, 1), (7, -3, 5)\}$ is linearly independent in $\mathbb{R}^3$ (5)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

(a) Use Gaussian elimination to solve the following system

$$\begin{aligned} x - 2y + z - 4w &= 1 \\ x + 3y + 7z + 2w &= 2 \\ x - 12y - 11z - 16w &= -1 \end{aligned}$$

(b) Use the augmented matrix $[A : I]$ to find A^{-1} where $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ (4)

(c) Use (b) to solve $A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$ (4)

(d) Use (b) to write A^{-1} and A as a product of elementary matrices (4)

QUESTION B4 [*20 Marks*]

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(a) State the Cayley-Hamilton theorem (do not prove anything) (3)

(b) Verify the Cayley-Hamilton theorem with $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ (5)

(c) Let B_1 and B_2 be finite subsets of a vector space V and let B_1 be a subset B_2. Show that

(i) if B_1 is linearly dependent, B_2 is also linearly dependent.

(ii) if B_2 is linearly independent, B_2 is also linearly independet.

QUESTION B5 [*20 Marks*]

(a) For the matrix $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$ find its eigenvalues and the corresponding eigenvectors. (10)

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be $T(x, y) = (x-2y, 2x+y, x+y)$ Find the matrix of T with respect to B_1 and B_2 where $B_1 = \{(1, -1), (0, 1)\}$ $B_2 = \{(1, 1, 0), (0, 1, 1), (1, -1, 1)\}$ (10)

QUESTION B6 [*20 Marks*]

(a) Evaluate the following determinant by expanding along the second row

$$\begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & 2 & 2 \end{vmatrix}$$ (5)

(b) Find the co-ordinate vector of $u = (4, 0, -2)$ relative to the basis $B = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ (5)

(c) Consider the linear system

$$\begin{aligned} x - 2y + 3z &= 1 \\ 2x + ky + 6z &= 6 \\ -x + 3y + (k-3)z &= 0 \end{aligned}$$

Find values of k for which the linear system has

(i) a unique solution

(ii) no solution

(iii) infinitely many solutions (10)

QUESTION B7 [*20 Marks*]

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(a) Find the co-ordinate vector of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with respect to the basis

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$ (8)

(b) Prove that if a homogeneous system of linear equations has more unknowns than the number of equations than it has a non-trivial solution. (12)

END OF EXAMINATION PAPER