UNIVERSITY OF SWAZILAND

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SUPPLEMENTARY EXAMINATION, 2013/2014

B.Sc. II, BASS II, BED. II

Title of Paper : Linear Algebra

Course Number : M220

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked**.
2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

A1. (a) (i) Give the definition of a basis of a vector space

(ii) Determine whether the vectors $u_1 = (1,1,1)$ $u_2 = (1,2,3)$ $u_3 = (2,-1,1)$ form a basis for $\mathbb{R}^3$ (10)

(b) Let $S = \{v_1, v_2, \cdots, v_n\}$ be a set of non-zero vectors in a vector space V. Prove that S is linearly dependent if and only if one of the vectors v_j is linear combination of the proceeding vectors in S. (10)

A2. (a) (i) State the Cayley-Hamilton Theorem

(ii) Illustrate the validity of the Cayley-Hamilton Theorem using the matrix

$A = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$ (8)

(b) Prove that if a homogeneous system has more unknowns than the number of equations then it always has a non-trivial solution. (8)

(c) By inspection, find the inverses of the following elementary matrces

(i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ (4)

(ii) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (4)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

B3. (a) Determine whether the following homogeneous system has non-trivial solutions

$x_1 - 2x_2 + 3x_3 - x_4 = 0$

$3x_1 + x_2 - x_3 + 2x_4 = 0$

$2x_2 - x_1 + 2x_3 + 2x_4 = 0$ (3)

(b) Evaluate the determinant using cofactor expansion along the second row

$\begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & 2 & 2 \end{vmatrix}$ (4)

(c) (i) Find the inverse of the matrix A using the Gaussian elimination algorithm on $[A : I_4]$, and then use A^{-1} to solve the system $A \cdot X = B$ where

$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & 5 & -3 \\ 0 & -1 & 3 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} . \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(ii) Find a finite sequence of elementary matrices $E_1, E_2, \cdots, E_k$ such that $E_k \cdot E_{k-1} \cdots E_1 \cdot A = I_4$ (13)

QUESTION B4 [*20 Marks*]

B4. Lt $B = \{u_1, u_2, u_3\}$ and $B^1 = \{v_1, v_2, v_3\}$ bases in $\mathbb{R}^3$, where

$u_1 = (1,0,0)^T \quad y_2 = (1,1,0)^T \quad y_3 = (1,1,1)^T$

$v_1 = (0,2,1)^T \quad v_2 = (1,0,2)^T \quad v_3(1,-1,0)^T$

(a) Find the transition matrix from B^1 to B (7)

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the basis B is

$$[T]_B = \begin{bmatrix} 3 & -6 & 9 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

Find the matrix of T w.r. to B^1 (8)

(c) Let $u \in \mathbb{R}^3$ be the vector whose coordinates relative to B are

$$[u]_B = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}.$$

Find the coordinates of u relative to B^1 (5)

QUESTION B5 [*20 Marks*]

B5. (a) Solve the following system of linear equations

$x_1 - 2x_2 + 3x_3 - 4x_4 = 1$
$2x_1 + 5x_2 - 8x_3 + 6x_4 = 4$
$x_1 - 4x_2 - 7x_3 + 2x_4 = 8$ (8)

(b) For which k does the following system have nontrivial solutions

$kx_1 + 2x_2 - x_3 = 0$
$(k+1)x_1 + kx_2 + 0x_3 = 0$
$-x_1 + kx_2 + kx_3 = 0$ (8)

(c) Determine whether the given vectors are linearly independent

$u_1 = (2,4,0,4,3)^T \quad u_2(1,2,-1,3,1)^T \quad u_3 = (-1,-2,5,-7,1)^T$ (4)

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QUESTION B6 [*20 Marks*]

B6. (a) Use Cramer's rule to solve the following system

2x + 3y - z = 1
3x + 5y+2z= 8
x-2y-3z=-1 (7)

(b) Compute $det(A)$ and, if A is invetible, firn $det(A^{-1})$, where

$$A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 272 & -1 & 1 & \end{bmatrix}$$ (6)

(c) Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$

Use the adjoint of A to find A^{-1} (7)

(5)

QUESTION B7 [*20 Marks*]

B7. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 2y \\ 2x + 2y \\ x + y \end{pmatrix}$$

Find the matrix of T

(i) with respect to the standard basis

(ii) with respect to B^1 and B where

$$B^1 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ and}$$

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$ (10)

(b) Let V be all ordered pairs of real numbers. Define addition and scale multiplication as follows

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and

$\alpha(x_1, y_1) = (\alpha x_1 + \alpha - 1, \alpha y_1 + \alpha - 1)$

show that V is vector space (10)

END OF EXAMINATION PAPER