UNIVERSITY OF SWAZILAND

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FINAL EXAMINATIONS 2013/2014

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ALL QUESTIONS IN SECTION A. 3. ANSWER ANY THREE QUESTIONS IN SECTION B.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

SECTION A

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QUESTION 1

Define each of the following mathematical terms:

(a) Axiom; [2]

(b) Conjecture; [2]

(c) Lemma; [2]

(d) Corollary; [2]

(e) Tautology; [2]

(f) Valid statement; [2]

(g) Argument; [2]

(h) Valid argument; [2]

(i) Invalid argument; [2]

(j) Mathematical proof. [2]

QUESTION 2

(a) What do you understand by the following?

- (i) Universal Instantiation; [3]
- (ii) Universal Generalization; [3]
- (iii) Existential Instantiation; [3]
- (iv) Existential Generalization. [3]

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(b) State the Pigeonhole Principle. [2]

(c) State the Principle of Mathematical Induction II. [2]

(c) State the Principle of Strong Mathematical Induction. [2]

(d) State the Fundamental Theorem of Arithmetic. [2]

SECTION B

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QUESTION 1

(a) Write down symbolically, the negation of the statements:

(i) $\exists x, (\neg P(x) \vee Q(x))$; [4]

(ii) $\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}\, \exists z \in \mathbb{R}, x^2 + y^2 < z$. [6]

(b) Let $A = \{-3, -2, -1, 0, 1, 2, 3\}$, where $A \subseteq \mathbb{R}$. Determine the truth set of

$$(\forall y \in A), x + y < 5.$$

[5]

(c) Determine the truth value in $\mathbb{R}$ of:

(i) $\exists x \in \mathbb{R}$ such that $|x| = -x$; [2]

(ii) $\exists x \in \mathbb{R} : x + 4 = x$. [3]

QUESTION 2

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(a) Write down the negation of the following statement: "The function f of one variable is a *convex function* if and only if for all real numbers x and y and for all real numbers t with $0 \leq t \leq 1$, it follows that $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$." [6]

(b) For each of the following, write the *converse* and the *contrapositive*:

(i) If n is an integer for which n^2 is even, then n is even. [3]

(ii) Suppose that t is an angle between 0 and π. If t satisfies $\sin(t) = \cos(t)$, then $t = \frac{\pi}{4}$. [3]

(c) Show that the proposition

$$[(P \implies Q) \wedge (Q \implies R)] \implies [P \implies R],$$

where P, Q and R are statements, is a tautology. [8]

QUESTION 3

(a) Prove that between any two *distinct irrational* numbers, there is a rational number and an irrational number. [10]

(b) Define the following:

(i) Fallacy of affirming the conclusion; [2]

(ii) Fallacy of denying the antecedent. [2]

(c) Using truth tables, analyze the following argument and state whether it is valid or invalid

"All Germans are Europeans.

My neighbor is not a German.

Therefore my neighbor is not a European." [6]

QUESTION 4

(a) Describe a modified induction procedure that could be used to prove statements of the form:

(i) For all integers $n \leq k$, $P(n)$ is true, where $P(n)$ is a statement containing the integer n. [3]

(ii) For all integers n, $P(n)$, where $P(n)$ is as in Part (i). [4]

(iii) For every positive odd integer, something happens. [3]

(b) For all non-negative integers n define the number u_n inductively as

$$\begin{aligned} u_0 &= 0, \\ u_{k+1} &= 3u_k + 3^k \qquad \text{for } k \geq 0. \end{aligned}$$

Prove that $u_n = n3^{n-1}$ for all non-negative integers n. [4]

(c) If $f(n) = 3^{2n} + 7$, where n is a natural number, show that $f(n+1) - f(n)$ is divisible by 8. Hence prove by induction that $3^{2n} + 7$ is divisible by 8. [6]

QUESTION 5

(a) Four intelligent frogs sit on a log; two green frogs on one side and two brown frogs on the other side, with an empty seat separating them. They decide to switch places. The only moves permitted are to jump over one frog of a different color into an empty space or to jump into an adjacent space. What is the minimum number of moves? Generalize this problem and solve it. [10]

(b) Using the axioms given below, prove each of the theorems which follow.

Axiom 1 All mathematicians are logical.

Axiom 2 Careful people are not foolish.

Axiom 3 Discontented people are foolish.

Axiom 4 Logical people are careful.

Theorem 1 *Mathematicians are contented.*

[4]

Theorem 2 *Foolish people are not logical.*

[3]

Theorem 3 *Careless people are not mathematicians.*

[3]

END OF EXAMINATION