# UNIVERSITY OF SWAZILAND

74

## SUPPLEMENTARY EXAMINATIONS 2013/2014

## B.Sc. / B.Ed. / B.A.S.S. II

| | | |
|---|---|---|
| TITLE OF PAPER | : | DYNAMICS I |
| COURSE NUMBER | : | M255 |
| TIME ALLOWED | : | THREE (3) HOURS |
| INSTRUCTIONS | : | 1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.<br>2. ANSWER ALL QUESTIONS IN SECTION A.<br>3. ANSWER ANY THREE QUESTIONS IN SECTION B. |
| SPECIAL REQUIREMENTS | : | NONE |

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

## SECTION A

### QUESTION 1

(a) Prove that the diagonals of a parallelogram bisect each other. [10]

(b) Decompose the vector $\mathbf{u} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ into vectors parallel and orthogonal to the vector $\mathbf{v} = -5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$. [6]

(c) Given the points $P(2, 1, -1)$, $Q(3, 0, 2)$, $R(4, -2, 1)$, and $S(5, -3, 0)$. find the volume of the parallelepiped having adjacent sides $PQ$, $PR$, and $PS$. [4]

### QUESTION 2

(a) Evaluate $\lim_{t\to\infty}\left(\mathrm{e}^{-t}\hat{\mathbf{i}} + \frac{1}{t}\hat{\mathbf{j}} + \frac{t}{t^2-1}\hat{\mathbf{k}}\right)$. [4]

(b) Let $\mathbf{F}(\mathbf{t}) = (\mathbf{e^t}\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \mathbf{t^2}\,\hat{\mathbf{k}}) \times (\mathbf{t^3}\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$. Find $\mathbf{F}'(t)$. [4]

(c) Prove that if $\mathbf{r}(t)\cdot\mathbf{r}(t)$ is a constant, then $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$. [3]

(d) Given that $\mathbf{F}(x, y, z) = \mathrm{e}^{xy}\hat{\mathbf{i}} + (x - y)\hat{\mathbf{j}} + zy\sin y\,\hat{\mathbf{k}}$, find $\mathbf{F}_x$, $\mathbf{F}_y$, $\mathbf{F}_z$, $\mathbf{F}_x \times \mathbf{F}_y$, and $\mathbf{F}_x \cdot \mathbf{F}_z$.

## SECTION B

75

### QUESTION 3

A parametrization of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \qquad a, b > 0,$$

traversed in the counterclockwise direction is given by $x = a\cos t$, $y = b\sin t$; $0 \leq t < \infty$. Suppose that a particle moves along this ellipse in the counterclockwise direction. Find:

(a) the position vector $\mathbf{r}$; [1]

(b) the velocity vector $\mathbf{v}$; [1]

(c) the speed $|\mathbf{v}|$; [1]

(d) the acceleration vector $\mathbf{a}$; [1]

(e) the magnitude of the acceleration $|\mathbf{a}|$; [1]

(f) the unit tangent vector $\hat{\mathbf{T}}$; [2]

(g) the principal unit normal vector $\hat{\mathbf{N}}$; [8]

(h) the curvature $\kappa$; [2]

(i) the unit binormal vector $\hat{\mathbf{B}}$; [3]

of the particle at the point $P\left(\frac{a}{\sqrt{2}}, \frac{-b}{\sqrt{2}}\right)$.

## QUESTION 4

(a) The force acting on a particle of mass $m$ is given by $\mathbf{F} = a\cos(\omega t)\hat{\mathbf{i}} + b\sin(\omega t)\hat{\mathbf{j}}$. If the particle is initially at rest at the origin, prove that its position at any later time $t$ is

$$\mathbf{r} = \frac{a}{m\omega^2}\Big(1 - \cos(\omega t)\Big)\hat{\mathbf{i}} + \frac{b}{m\omega^2}\Big(\omega t - \sin(\omega t)\Big)\hat{\mathbf{j}}.$$

[6]

(b) A lift ascends 400 meters in 2 minutes, traveling from rest to rest. It travels with uniform acceleration for the first 30 seconds; it travels with uniform retardation for the next 20 seconds; and it travels with uniform speed for the remainder of the time. Calculate:

(i) the uniform speed in meters per second; [4]

(ii) the uniform acceleration in meters per second squared; [5]

(iii) the time taken by the lift to ascend the first 200 meters. [5]

## QUESTION 5

78

(a) A particle moves on the $x$ axis, attracted to the origin $O$ by a force proportional to its distance from $O$. If the particle starts from rest at $x = 5$ cm and reaches $x = 2.5$ cm for the first time after 2 seconds, find:

(i) the position at any time $t$ after it starts;

(ii) the magnitude of the velocity at $x = 0$;

(iii) the amplitude, period, and frequency of the vibration; and

(iv) the acceleration. [6,3,2,1]

(b) (i) A 7 kg weight suspended at the end of a vertical spring stretches it 5 cm. Assuming that a damping force numerically equal to 0.2 times the velocity is acting on the system, find the position of the weight at any time $t$ if initially the weight is pulled down 10 cm and released.

(ii) Is the motion in (i) oscillatory, over-damped, or critically damped? [7,1]

## QUESTION 6

(a) A particle is projected vertically upwards with initial speed $u$. Gravity acts, as does air resistance, which is given by $kv$ per unit mass, where $k$ is a constant and $v$ is the speed of the particle. Find the time taken to reach the maximum height. [10]

(b) A particle is projected with velocity $\mathbf{u}$ from a point $O$ in a vertical plane through the line of greatest slope of a plane inclined at an angle $\beta$ to the horizontal. After time $T$, the particle strikes the inclined plane at the point $P$, at a distance $R$ from $O$. If $\mathbf{u}$ makes an angle $\alpha$ with the horizontal, and if $|\mathbf{u}| = u$, show that:

(i) $T = \dfrac{2u\sin(\alpha - \beta)}{g\cos\beta}$ and $R = \dfrac{u^2[sin(2\alpha - \beta) - \sin\beta]}{g\cos^2\beta}$;

(ii) for constant $u$ and $\beta$, $R$ is maximum when $\alpha = \dfrac{\pi}{4} + \dfrac{\beta}{2}$. [8,2]

## QUESTION 7

(a) If $\mathbf{r} = \mathbf{r}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, $r = |\mathbf{r}|$, and $\mathbf{u} = \mathbf{u}(x, y, z,) = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$ is any vector whose first partial derivatives exist, show that:

(i) $\nabla\cdot\mathbf{r} = 3$; [2]

(ii) $\nabla\times\mathbf{r} = \mathbf{0}$; [3]

(iii) $\nabla\cdot(\nabla\times\mathbf{u}) = 0$; [4]

(iv) $\nabla r = \dfrac{\mathbf{r}}{r}$. [3]

(b) A particle of mass $m$ moves in a central force field $\mathbf{F} = \frac{K}{r^n}\hat{\mathbf{r}}$ where $K$ and $n$ are constants. It starts from rest at $r = a$ and arrives at $r = 0$ with finite speed $v_0$. Prove that

$$v_0 = \left\{\frac{2Ka^{1-n}}{m(n-1)}\right\}^{1/2}.$$

[8]

END OF EXAMINATION

## QUESTION 6

(a) A particle is projected vertically upwards with initial speed $u$. Gravity acts, as does air resistance, which is given by $kv$ per unit mass, where $k$ is a constant and $v$ is the speed of the particle. Find the time taken to reach the maximum height. [10]

(b) A particle is projected with velocity $\mathbf{u}$ from a point $O$ in a vertical plane through the line of greatest slope of a plane inclined at an angle $\beta$ to the horizontal. After time $T$, the particle strikes the inclined plane at the point $P$, at a distance $R$ from $O$. If $\mathbf{u}$ makes an angle $\alpha$ with the horizontal, and if $|\mathbf{u}| = u$, show that:

(i) $T = \dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$ and $R = \dfrac{u^2[sin(2\alpha-\beta)-\sin\beta]}{g\cos^2\beta}$;

(ii) for constant $u$ and $\beta$, $R$ is maximum when $\alpha = \dfrac{\pi}{4}+\dfrac{\beta}{2}$. [8,2]

## QUESTION 7

(a) If $\mathbf{r} = \mathbf{r}(x,y,z) = x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, $r = |\mathbf{r}|$, and $\mathbf{u} = \mathbf{u}(x,y,z,) = u_1\hat{\mathbf{i}}+u_2\hat{\mathbf{j}}+u_3\hat{\mathbf{k}}$ is any vector whose first partial derivatives exist, show that:

(i) $\nabla\cdot\mathbf{r} = 3$; [2]

(ii) $\nabla\times\mathbf{r} = \mathbf{0}$; [3]

(iii) $\nabla\cdot(\nabla\times\mathbf{u}) = 0$; [4]

(iv) $\nabla r = \dfrac{\mathbf{r}}{r}$. [3]

(b) A particle of mass $m$ moves in a central force field $\mathbf{F} = \frac{K}{r^n}\hat{\mathbf{r}}$ where $K$ and $n$ are constants. It starts from rest at $r = a$ and arrives at $r = 0$ with finite speed $v_0$. Prove that

$$v_0 = \left\{\frac{2Ka^{1-n}}{m(n-1)}\right\}^{1/2}.$$

[8]

END OF EXAMINATION