UNIVERSITY OF SWAZILAND

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FINAL EXAMINATIONS 2013/2014

B.Sc. / B.Ed. / B.A.S.S. / B.EENG. III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ALL QUESTIONS IN SECTION A. 3. ANSWER ANY THREE QUESTIONS IN SECTION B.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

SECTION A

QUESTION 1

(a) Give a formula $\mathbf{F} = M(x,y)\hat{\mathbf{i}} + N(x,y)\hat{\mathbf{j}}$ for the vector field in the plane with the properties that $\mathbf{F} = \mathbf{0}$ at the origin and that at any other point (a,b) in the plane, $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction, with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$. [8]

(b) Find the tangent plane and the normal line to the surface $x^2y + xyz - z^2 = 1$ at the point $P_0(1,1,3)$. [7]

(c) Determine the directional derivative of $\phi(x,y) = \ln\sqrt{x^2+y^2}$ at the point $(1,0)$ in the direction of $\dfrac{2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}}{2\sqrt{2}}$. [5]

QUESTION 2

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(a) Show that $\mathbf{n}(t) = -g'(t)\hat{\mathbf{i}} + f'(t)\hat{\mathbf{j}}$ and $-\mathbf{n}(t) = g'(t)\hat{\mathbf{i}} - f'(t)\hat{\mathbf{j}}$ are both normals to the curve $\mathbf{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ at the point $(f(t), g(t))$. Hence find a unit normal, $\hat{\mathbf{N}}$, for the curve $\mathbf{r}(t) = \sqrt{4-t^2}\,\hat{\mathbf{i}} + t\hat{\mathbf{j}}, \quad -2 \le t \le 2.$ [8]

(b) Suppose that a curve in the xy-plane is the graph of $y = f(x)$. Show that the arc-length, $s = s(x)$ from $x = a$ to $x = b$ is given by

$$s = \int_a^b \left[1 + (f'(x))^2\right]^{\frac{1}{2}} \mathrm{d}\,x.$$

[3]

(c) Find the arc length parameter s along the curve $\mathbf{r}(t) = (4\cos t)\hat{\mathbf{i}} + (4\sin t)\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}$; $t \ge 0$, from the point $t = 0$ by first evaluating the integral

$$s = \int_{\tau=0}^{t} |\mathbf{v}(\tau)|\mathrm{d}\tau.$$

Then find the length of the portion of the curve between $t = 0$ and $t = \frac{\pi}{2}$. [4]

(d) Reparametrize the curve $x = \mathrm{e}^t\cos t,\ y = \mathrm{e}^t\sin t,\ z = \mathrm{e}^t;\ \ -\infty < t < \infty$ by the arc-length parameter s. [5]

SECTION B

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QUESTION 3

Let PQR be a triangle with vertices $P(0,0,0), Q(2b_1,0,0)$, and $R(2c_1,2c_2,2c_3)$, where $b_1, c_1, c_2, c_3 \in \mathbb{R}$. Furthermore, assume that $b_1 \neq 0$, that c_2 and c_3 are not simultaneously 0, and let $S = S\left(\frac{2b_1+2c_1}{3}, \frac{2c_2}{3}, \frac{2c_3}{3}\right)$ be a point inside Triangle PQR. Show that the medians of Triangle PQR intersect at S. Hence, or otherwise, prove that the medians of a triangle intersect at a single point. [20]

QUESTION 4

(a) Find parametric equations for the line that is tangent to the curve traced by $\mathbf{r}(t) = (a\sin t)\hat{\mathbf{i}} + (a\cos t)\hat{\mathbf{j}} + bt\hat{\mathbf{k}};\quad t \geq 0$ at the point $P(t_0)$ when $t_0 = 4\pi$. [5]

(b) Let $\mathbf{u}(t) = \frac{1}{t}\hat{\mathbf{i}} - \hat{\mathbf{j}} + \ln t\hat{\mathbf{k}}$ and $\mathbf{v}(t) = t^2\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + \hat{\mathbf{k}}$. Find;

(i) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)\cdot\mathbf{v}(t))$; [3]

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)\times\mathbf{u}'(t))$. [3]

(c) Given that $\mathbf{F}(t) = \mathrm{e}^{2t}\mathbf{u} + \mathrm{e}^{3t}\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are constant vectors, show that $\mathbf{F}''(t) - 5\mathbf{F}'(t) + 6\mathbf{F}(t) = \mathbf{0}$. [6]

(d) Evaluate $\lim_{t\to\infty}\left(\mathrm{e}^{-t}\hat{\mathbf{i}} + \frac{1}{t}\hat{\mathbf{j}} + \frac{t}{t^2-1}\hat{\mathbf{k}}\right)$. [3]

QUESTION 5

(a) Part of a railway line (superimposed on a rectangular coordinate system) follows the line $y = -x$ for $x \leq 0$, then turns to reach the point (6,0) following a cubic curve. Find the equation of this curve if the track is continuous, smooth, and has continuous curvature. [12]

(b) Find the scale factors h_1, h_2 and h_3 in cylindrical and in spherical coordinates. Hence find the volume element $\mathrm{d}V$ in cylindrical and in spherical coordinates. [8]

QUESTION 6

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(a) Integrate $f(x, y, z) = 2x - 6y^2 + 2z$ over the line segment C joining the points (2,2,2) and (3,3,3). [5]

(b) Find the work done in moving a particle in the counterclockwise direction once around the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ if the force field is given by $\mathbf{F} = (3x - 4y)\hat{\mathbf{i}} + (4x + 2y)\hat{\mathbf{j}} - 4y^2\hat{\mathbf{k}}$. [6]

(c) Show that $y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z$ is exact, and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z.$$

[9]

QUESTION 7

(a) For the vector field $\mathbf{F}(x, y, z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, find the flow line through the point $P(2, 2, 1)$. [6]

(b) Find the flux of $\mathbf{F} = yz\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} - z^2\,\hat{\mathbf{k}}$ outward through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$. [8]

(c) Verify Stoke's theorem for the vector field $\mathbf{F}(x, y, z) = (2x - y)\,\hat{\mathbf{i}} - yz^2\,\hat{\mathbf{j}} - y^2z\,\hat{\mathbf{k}}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane and C is its boundary. [6]

END OF EXAMINATION