UNIVERSITY OF SWAZILAND

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SUPPLEMENTARY EXAMINATIONS 2013/2014

B.Sc. / B.Ed. / B.A.S.S. / B.EENG. III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ALL QUESTIONS IN SECTION A. 3. ANSWER ANY THREE QUESTIONS IN SECTION B.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

SECTION A

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QUESTION 1

(a) (i) The graph $y = f(x)$ in the xy-plane automatically has the parametrization $x = x$, $y = f(x)$, and the vector formula $\mathbf{r}(x) = x\hat{\mathbf{i}} + (f(x))\hat{\mathbf{j}}$. Use this formula to show that if f is a twice differentiable function of x, then

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

[7]

(ii) Use the formula for κ in (i) to find the curvature of $y = \ln(\cos x)$, $-\pi/2 \leq x \leq \pi/2$. [3]

(b) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in space. Prove the *Pythagorean Principle,*

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 \iff \mathbf{u} \cdot \mathbf{v} = 0.$$

[6]

(c) Find parametric equations for the tangent line to the curve $C_3 : x = -t - 8,\quad y = t^2 - 3,\quad z = 2t - 5;\quad -\infty < t < \infty$, at the point $(-9, -2, -3)$. [4]

QUESTION 2

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(a) A path of a roller coaster ride (superimposed on a rectangular coordinate system) consists of part of the parabola $y = x^2/2$ for $x \leq 0$, followed by a circular loop for $x \geq 0$. Find the equation of this loop if the track is *continuous*, *smooth*, and has *continuous curvature.* [8]

(b) Let $\mathbf{F} = (6xy + z^3)\hat{\mathbf{i}} + (3x^2 - z)\hat{\mathbf{j}} + (3xz^2 - y)\hat{\mathbf{k}}$ be a vector field.

(i) Show that $\mathbf{F}$ is irrotational. [3]

(ii) Find $\operatorname{div}\operatorname{curl}\mathbf{F}$. [2]

(c) Let $\mathbf{u}(x, y, z) = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$ and $\mathbf{v}(x, y, z) = \dfrac{\mathbf{u}}{(x^2 + y^2)^{\frac{1}{2}}}$ be vectors in space. Find the flow lines of $\mathbf{u}$ and $\mathbf{v}$. [5,2]

SECTION B

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QUESTION 3

(a) Find the principal unit normal vector and the outward unit normal vector to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \qquad a, b > 0,$$

traversed in the clockwise direction, at the point point $P\left(\frac{a}{\sqrt{2}}, \frac{-b}{\sqrt{2}}\right)$. Also, find the curvature, κ, and the radius of curvature, ρ, at the given point. [14]

(b) Find parametric equations for the line of intersection of the planes $3x-6y-2z = 15$ and $2x + y + -2 = 5$. [6]

QUESTION 4

(a) Derive the following alternative formula for the curvature function $\kappa = \kappa(t)$ of a smooth curve:

$$\kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3}.$$

[6]

(b) Let $\mathbf{r}(t) = 6\cos t\hat{\mathbf{i}} + 6\sin t\hat{\mathbf{j}} + 2t\hat{\mathbf{k}}$. Find the following:

(i) $\mathbf{r}'(t)\cdot\mathbf{r}''(t)$; [2]

(ii) $\mathbf{r}'(t)\times\mathbf{r}''(t)$. [3]

(c) Evaluate $\lim\limits_{t\to 1}\left(\frac{1}{t}\hat{\mathbf{i}} + \frac{\ln t}{t^2-1}\hat{\mathbf{j}} + \frac{t-1}{t^2-1}\hat{\mathbf{k}}\right)$. [3]

(d) Prove the *Cauchy-Schwarz Inequality*; $|\mathbf{u}\cdot\mathbf{v}| \le |\mathbf{u}||\mathbf{v}|$, where $\mathbf{u}$ and $\mathbf{v}$ are vectors in space. [2]

(e) Is the line $x = 1-2t, \quad y = 2+5t, \quad z = -3t$ parallel to the plane $2x+y-z = 8$? Give reasons for your answer. [4]

QUESTION 5

(a) Let D be the region in the xyz-space defined by the inequalities

$$1 \leq x \leq 2, \qquad 0 \leq xy \leq 2, \qquad 0 \leq z \leq 1.$$

Evaluate

$$\iiint_D (x^2y + 3xyz)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

by applying the transformation

$$u = x, \qquad v = xy, \qquad w = 3z$$

and integrating over the appropriate region G in the uvw-space. [10]

(b) Find out which of the fields given below are conservative. For conservative fields, find a potential function.

(i) $\mathbf{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$. [2]

(ii) $\mathbf{F} = \frac{x}{x^2+y^2+z^2}\,\hat{\mathbf{i}} + \frac{y}{x^2+y^2+z^2}\,\hat{\mathbf{j}} + \frac{z}{x^2+y^2+z^2}\,\hat{\mathbf{k}}$. [8]

QUESTION 6

(a) By any method, find the integral of $H(x, y, z) = x^2z$ over the surface of the sphere $x^2 + y^2 + z^2 = 1;\ z \geq 0$ [10]

(b) Verify Green's theorem in the plane for

$$\oint_C [(2xy - x^2)\mathrm{d}x + (x + y^2)\mathrm{d}y],$$

where C is the closed curve (described in the positive direction) of the region bounded by the curves $y = x^2$ and $y^2 = x$. [10]

QUESTION 7

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(a) If $\mathbf{F} = y\hat{\mathbf{i}} + (x - 2xz)\hat{\mathbf{j}} - xy\hat{\mathbf{k}}$, evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, \mathrm{d}S$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane. [10]

(b) Verify that the parametric equations

$$x = \rho^2 \cos\theta, \quad y = \rho^2 \sin\theta, \quad z = \rho$$

could be used to represent the surface $x^2 + y^2 - z^4 = 0$. Hence compute the unit normal to this surface at any point. [10]

END OF EXAMINATION