UNIVERSITY OF SWAZILAND

FINAL EXAMINATION, 2013/2014

B.Sc. III, B.Eng III, B.Ed III, BASS III

Title of Paper : Abstract Algebra I

Course Number : M323

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked**.
2. Show all your working.

Special Requirements: None

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: Answer ALL Questions ,08

A1. (a) Find (a, b) and $[a, b]$ by first decomposing (writing) as a product of primes

$$a = 144 \qquad b = 1250$$

(10)

(b) Solve the system

$$\begin{aligned} 3x &\equiv 2(mod5) \\ 2x &\equiv 1(mod3) \end{aligned}$$

A2. Let $\varphi : G \to H$ be an isomorphism of groups.

(i) Prove that if e_G and e_H are the identity elements of G and H respectively, then

(i) $(e_G)\varphi = e_H$

(ii) $[(a)\varphi]^{-1} = (a^{-1})\varphi \forall a \in G$. (6)

(b) Given an example of a group satisfying the given conditions or, if there is no such group, say so. (Do not prove anything)

(i) A cyclic group of order 4

(ii) A non-abelian group of order 5

(iii) An infinite cyclic group

(iv) A non-abelian cyclic group (8)

SECTION B: Answer any *THREE* Questions

QUESTION B1 [*20 Marks*]

B1. (a) Prove that if G is a group and that $\forall a \in G \;\; a^2 = e$, then G is abelian. (10)

(b) For a group G define the following relation for $a, b, \in G$:
"$aRb \Leftrightarrow$ there exists $x \in G$ such tat $b = x^{-1}ax$"
Show that the above relation is an equivalence relation (10)

QUESTION B2 [*20 Marks*]

B2. (a) (i) Find all the conjugate elements of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ is S_3 (5)

(ii) Determine the order of (1346)(287) in S_8 (5)

(b) Prove that every group of prime order is cyclic. (10)

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QUESTION B3 [*20 Marks*]

B3. (a) Find all subgroups of $\mathbb{Z}_{12}$ and draw the lattice diagram (10)

(b) For each binary operation $\star$ defined on a set G, say whether or not $\star$ gives a group structure on the set

(i) $G = \mathbb{Q}$; $a \star b = \frac{a \times b}{3}$

(ii) $G = \mathbb{Q} - \{1\}$ $a \star b = a + b + a \times b$

where: $+$ is the usual addition and

$\times$ is the usual multiplication. (3)

QUESTION B4 [*20 Marks*]

B4. Let $\alpha \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 7 & 8 & 5 & 6 & 4 \end{pmatrix}$ and $\beta \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 7 & 6 & 2 & 8 & 4 & 3 \end{pmatrix}$

(a) Express α and β as products of disjoint cycles, and then as products of transpositions. For each of them, say whether it is an even permutation or an odd one. (6)

(b) Compute α^{-1}, $\beta^{-1} o \alpha$ and $(\alpha o \beta)^{-1}$ (9)

(c) Find the order of β. (5)

QUESTION B5 [*20 Marks*]

B5. (a) Let $H =< 8 >$ be the group of $\mathbb{Z}_{20}$ generated by the element 8. Find all cosets of H in $\mathbb{Z}_{20}$ (10)

(b) Prove that every subgroup of a cyclic group is cyclic. (10)

END OF EXAMINATION PAPER