UNIVERSITY OF SWAZILAND

M 331

MAIN EXAMINATION 2013/2014

BSc. /BEd. /B.A.S.S II

TITLE OF PAPER	:	Real Analysis
COURSE NUMBER	:	M 331
TIME ALLOWED	:	3 HOURS
SPECIAL REQUIREMENTS	:	NONE

Instructions

(a) Candidates may attempt:

- (i) ALL questions in Section A and
- (ii) At most THREE questions in Section B.

(b) Each question should start on a fresh page.

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A6.

A1. (a) Find all $x \in \mathbb{R}$ that satisfy the inequality $|x+1| + |x+2| < 3$. [4]

(b) What does it mean to say that $\alpha \in \mathbb{R}$ is an upper bound for a non-empty set $S \subseteq \mathbb{R}$? [2]

(c) Does the set $\{x \in \mathbb{R} : |x+1| + |x+2| < 3\}$ have an upper bound? Justify your answer. [2]

A2. (a) Precisely explain each of the following statements about a sequence (x_n) in $\mathbb{R}$.

(i) (x_n) is convergent. [2]

(ii) (x_n) is Cauchy. [2]

(b) State the Cauchy convergence criterion for sequences in $\mathbb{R}$. [2]

A3. (a) Precisely explain the followings statements about a function $f : [a, b] \to \mathbb{R}$.

(i) f is continuous at $c \in (a, b)$. [2]

(ii) f is continuous on $[a, b]$. [2]

(b) Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is not continuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$. Justify your answer. [2]

A4. Let $f : [a, b] \to \mathbb{R}$ be a function and let $c \in (a, b)$.

(a) Precisely explain the statement: "f is differentiable at c". [2]

(b) Is the following statement true or false? Justify your answer.
"f is differentiable at c whenever f is continuous at c". [2]

(c) Give an example of two functions $f, g : [-1, 1] \to \mathbb{R}$ that are both not differentiable at $x = 0$ but such that $f + g$ is differentiable at $x = 0$. Justify your answer. [2]

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A5. (a) Precisely explain the following statements about a series $\sum a_n$ in $\mathbb{R}$

(i) $\sum a_n$ is convergent. [2]

(ii) $\sum a_n$ is absolutely convergent. [2]

(b) State the Cauchy convergence criterion for series in $\mathbb{R}$. [2]

(c) Is the following statement true or false? Justify your answer.
"There is a series in $\mathbb{R}$ that is convergent but not absolutely convergent". [2]

A6. (a) State the Riemann criterion for a function $f : [a, b] \to \mathbb{R}$ to be integrable. [2]

(b) Is each of the following statements true or false? Justify your answers.

(i) There are 2 distinct functions $g, h : [0, 1] \to \mathbb{R}$ such that $g < h$ and yet $\int_0^1 g > \int_0^1 h$. [2]

(ii) There is a function $f : [0, 1] \to \mathbb{R}$ such that $|f|$ is integrable but f is not integrable. [2]

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SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B7 to B11.

B7. Let S be a non-empty set in $\mathbb{R}$ and let $u, v \in \mathbb{R}$.

(a) Precisely explain each of the following statements.

(i) $u = \sup S$. [2]

(ii) $v = \inf S$. [2]

(iii) S has a minimum in $\mathbb{R}$. [2]

(iv) S is bounded below. [2]

(b) Let $b < 0$ and let $T := \{bx : x \in S\}$. Prove that $\sup T = b \inf S$ and $\inf T = b \sup S$. [8]

(c) Prove that if S has a minimum in $\mathbb{R}$ then S is bounded below. [4]

B8. (a) Precisely explain each of the following statements about a sequence (x_n) in $\mathbb{R}$.

(i) (x_n) is bounded below. [2]

(ii) (x_n) is decreasing. [2]

(b) Prove that if (x_n) is bounded below and decreasing then it is convergent. [4]

(c) Let $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$.

(i) Show that (x_n) is bounded below by 2. [5]

(ii) Also show that (x_n) is decreasing. [5]

(iii) Find $\lim(x_n)$. [2]

B9. (a) Let $f, g : [a, b] \to \mathbb{R}$ be functions, let $k \in \mathbb{R}$ and let $c \in (a, b)$.

(i) Give a precise meaning of the statement: $\lim\limits_{x \to c} f$ exists. [2]

(ii) Show that if both $\lim\limits_{x \to c} f$ and $\lim\limits_{x \to c}(f + g)$ exist then $\lim\limits_{x \to c} g$ exists. [4]

(iii) Prove that $\lim\limits_{x \to c} kf = k \lim\limits_{x \to c} f$. [4]

(iv) Give an example of f, c for which $\lim\limits_{x \to c} f$ does not exist. [2]

(b) State (Bolzano's) Intermediate Value Theorem. [2]

(c) Use the Intermediate Value Theorem to show that equation $\sin x = 1 - x$ has a solution in the interval $[0, \pi/2]$. [4]

(d) Give an example of functions $f, g : [-1, 1] \to \mathbb{R}$ that are both not continuous at $x = 0$ and yet fg is continuous at $x = 0$. [2]

B10. (a) Prove that if a function $f : [a, b] \to \mathbb{R}$ is differentiable at $c \in (a, b)$ then f is continuous at c. [4]

(b) Is the converse of the previous statement true? Justify your answer. [2]

(c) Let $f : [-1, 1] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & \text{for } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that f is differentiable at $x = 0$. [4]

(ii) Is f continuous at $x = 0$? Justify your answer. [2]

(d) State the Mean Value Theorem for derivatives. [2]

(e) Suppose that $f : [0, 2] \to \mathbb{R}$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$ with $f(1) = f(2) = 1$. Show that $\exists c \in (1, 2) : f'(c) = 0$. [2]

(f) Use the Mean Value Theorem to prove that $0.5 < \ln 2 < 1$. [4]

B11. (a) Prove that if $\sum a_n$ converges then $\lim(a_n) = 0$. [4]

(b) Is the converse of the previous statement true? Justify your answer. [2]

(c) If $\sum a_n$ is absolutely convergent and (a_n) is a bounded sequence, show that $\sum a_n^2$ is absolutely convergent. [4]

(d) Let function $f : [0, 2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2 & \text{if } x \in [0, 1) \\ 3 & \text{if } x = 1 \\ 1 & \text{if } x \in (1, 2] \end{cases}$$

Show that f is integrable on $[0, 2]$ and evaluate its integral. [10]

END OF QUESTION PAPER