UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2013/2014

BSc. /BEd. /B.A.S.S II

TITLE OF PAPER	:	Real Analysis
COURSE NUMBER	:	M 331
TIME ALLOWED	:	3 HOURS
SPECIAL REQUIREMENTS	:	NONE

Instructions

(a) Candidates may attempt:

(i) ALL questions in Section A and

(ii) At most THREE questions in Section B.

(b) Each question should start on a fresh page.

1R

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A6.

A1. (a) Find all $x \in \mathbb{R}$ that satisfy the inequality $3x < |x-2|$. [4]

(b) State the infimum property for a non-empty subset of $\mathbb{R}$. [2]

(c) Does the set $\{x \in \mathbb{R} : |x+1| + |x+2| < 3\}$ have an infimum? Justify your answer. [2]

A2. Let $(x_n), (y_n)$ be sequences in $\mathbb{R}$.

(a) Give a precise meaning of the statement: "(x_n) converges". [2]

(b) Show that if both (x_n) and (y_n) converge then $(x_n - y_n)$ converges. [4]

A3. (a) Give a precise of the statement: "a function $f : [a, b] \to \mathbb{R}$ is is continuous at $c \in (a, b)$". [2]

(b) Show that if both functions $f, g : [0, 1] \to \mathbb{R}$ are continuous at $c \in (0, 1)$ then $f - g$ is continuous at c. [4]

A4. (a) Precisely explain the statement: "a real number α is the derivate of a function $f : [a, b] \to \mathbb{R}$". [2]

(b) State the mean value theorem for derivatives. [2]

(c) Give an example of two functions $f, g : [-1, 1] \to \mathbb{R}$ that are both not differentiable at $x = 0$ but such that fg is differentiable at $x = 0$. Justify your answer. [2]

A5. (a) Precisely explain the following statements about a series $\sum a_n$ in $\mathbb{R}$

(i) $\sum a_n$ is convergent. [2]

(ii) $\sum a_n$ is absolutely convergent. [2]

(b) Prove that if $\sum a_n$ converges then $\lim(a_n) = 0$. [4]

A6. Use the Riemann criterion to show that the function $f : [a, b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{when } x \in [2, 1) \\ 2 & \text{when } x = 1 \end{cases}$$

is integrable and $\int_0^1 f = 1$. [6]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B7 to B11.

B7. Let S be a non-empty bounded set in $\mathbb{R}$.

(a) Let $u, v \in \mathbb{R}$. Precisely explain the statement: "$u = \sup S$ and $v = \inf S$". [4]

(b) Let A, B be bounded non-empty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$. [8]

(c) Precisely explain each of the following statements.

(i) S has a maximum in $\mathbb{R}$. [2]

(ii) S is bounded above. [2]

(d) Prove that if S has a maximum in $\mathbb{R}$ then S is bounded above. [4]

B8. (a) Precisely explain each of the following statements about a sequence (x_n) in $\mathbb{R}$.

(i) (x_n) is bounded below. [2]

(ii) (x_n) is bounded above. [2]

(iii) (x_n) is decreasing. [2]

(iv) $\lim(x_n)$ exists. [2]

(b) Let $x_1 = 2$ and let $7x_{n+1} = 2x_n^2 + 3$ for $n = 1, 2, 3, \ldots$.

(i) Show that (x_n) is bounded below by $\frac{1}{2}$ and bounded above by 3. [5]

(ii) Also show that (x_n) is decreasing. [5]

(iii) Find $\lim(x_n)$. [2]

B9. (a) Let $f, g : [a, b] \to \mathbb{R}$ be functions, let $k \in \mathbb{R}$ and let $c \in (a, b)$.

(i) Give a precise meaning of the statement: $\lim_{x \to c} f$ exists. [2]

(ii) Show that if both $\lim_{x \to c} f$ and $\lim_{x \to c}(f + g)$ exist then $\lim_{x \to c} g$ exists. [4]

(iii) Show that if $\lim_{x \to c} f, \lim_{x \to c} g$ exist then $\lim_{x \to c} fg = \lim_{x \to c} f \lim_{x \to c} g$. [4]

(iv) Give an example of f, c for which $\lim_{x \to c} f$ does not exist. [2]

(b) State (Bolzano's) Intermediate Value Theorem. [2]

(c) Use the Intermediate Value Theorem to show that equation $\log x = 2 - x$ has a solution in the interval $[1, 2]$. [4]

(d) Give an example of distinct functions $f, g : [-1, 1] \to \mathbb{R}$ that are both not continuous at $x = 0$ and yet $f - g$ is continuous at $x = 0$. [2]

B10. (a) Let $f : [a, b] \to \mathbb{R}$ be a function, and let $c \in (a, b)$. Precisely explain each of the following statements.

(i) A real number α is the left derivative of f at c. [2]

(ii) A real number β is the right derivative of f at c. [2]

(b) Let $f : [-1, 1] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 - x & \text{for } x \geq 0 \\ e^{-x} & \text{otherwise} \end{cases}$$

(i) Show that f is differentiable at $x = 0$. [4]

(ii) Is f continuous at $x = 0$? Justify your answer. [2]

(c) State the Mean Value Theorem for derivatives. [2]

(d) Use the Mean Value Theorem to prove the following.

(i) $|\sin x| \leq |x|, \forall x \in \mathbb{R}$. [4]

(ii) The polynomial $p(x) = x^3 + ax + b$ (with $a > 0$) has exactly one real root. [4]

B11. (a) Prove that if a series in $\mathbb{R}$ is absolutely convergent then it is convergent. [4]

(b) Is the converse of the previous statement true? Justify your answer. [2]

(c) If $\sum a_n$ is absolutely convergent and (b_n) is a bounded sequence, then show that $\sum a_n b_n$ is absolutely convergent. [4]

(d) Let function $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2 & \text{if } x = 0 \\ 1 & \text{if } x \in (0, 1) \\ 3 & \text{if } x = 1 \end{cases}$$

Show that f is integrable on $[0, 1]$ and evaluate its integral. [10]

END OF QUESTION PAPER