130

UNIVERSITY OF SWAZILAND

M 411

MAIN EXAMINATION 2013/2014

BSc. /BEd. /B.A.S.S II

TITLE OF PAPER	:	Numerical Analysis II
COURSE NUMBER	:	M 411
TIME ALLOWED	:	3 HOURS
SPECIAL REQUIREMENTS	:	NONE. **NOT EVEN GRAPH IS REQUIRED.**

Instructions

(a) Candidates may attempt:

  (i) ALL questions in Section A and

  (ii) At most THREE questions in Section B.

(b) Each question should start on a fresh page.

(c) **THERE ARE NO SPECIAL REQUIREMENTS FOR THIS PAPER.**

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A3.

A1. (a) Let $w(x), \phi_0(x), \phi_1(x), \ldots, \phi_n(x)$ be functions that are defined on an interval $I \subseteq \mathbb{R}$. Precisely explain the following statements.

(i) $w(x)$ is a weight function on I. [2]

(ii) The set $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ is orthogonal on I with respect to weight function $w(x)$. [2]

(iii) $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ is linearly independent on I. [2]

(b) Suppose that $w(x)$ is a weight function on $[a, b]$, and that

$$\int_a^b w = 3, \quad \int_a^b w\,x = 2$$

(i) If $\phi_0(x) = 1$, determine polynomial $\phi_1(x)$ of degree 1 so that the set $\{\phi_0, \phi_1\}$ is orthogonal on $[a, b]$ with respect to weight w. [4]

(ii) Is $\{\phi_0, \phi_1\}$ linearly independent on $[a, b]$? Justify your answer. [2]

(iii) Given an example of a weight function on the interval $(0, \infty)$. Justify your answer. [2]

A2. An initial value problem consists of finding $y(t)$ satisfying

$$\frac{dy}{dt} = e^{t-y}, \; 0 \le t \le 1, \; y(0) = 1 \qquad (1)$$

(a) Use one step of each of the following methods to approximate $y(0.5)$.

(i) Taylor series method of order 2. [3]

(ii) Modified Euler method. [5]

(b) Let $f(t, y) = e^{t-y}$ and let $w_i \approx y(ih)$. A multi-step method for solving (1) with step size h is defined by

$$w_{i+2} = 4w_{i+1} - 3w_i - 2hf(t_i, w_i)$$

for each $i = 0, 1, \ldots, N-2$ where starting values w_0, w_1 are given. Show that the local truncation error for this method is $O(h^2)$. [6]

A3. (a) Let Γ be the boundary of an *open* and *connected* region $\Omega \subseteq \mathbb{R}^2$. A boundary value problem consists of finding $u(x, y)$ satisfying

$$\nabla^2 u = 0 \text{ on } \Omega,$$
$$u = f \text{ on } \Gamma$$

where $f(x, y)$ is a given function. Derive the *five point formula* for approximating u at any grid point in Ω. **Be careful to explain any notation used.** [5]

(b) Let a be a constant, let $j = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$. The upstream scheme

$$U_j^{n+1} = U_j^n - \alpha(U_j^n - U_{j-1}^n), \quad (2)$$

where $\alpha = a\frac{\Delta t}{\Delta x}$, can be used for approximating the advection equation

$$u_t + au_x = 0 \quad (3)$$

(i) Derive equation (2). [5]

(ii) Specify the range of values of α for which the upstream scheme converges. Justify your answer. [2]

SECTION B (60 marks)

Candidates may attempt THREE questions being careful to number them B4 to B8.

B4. Laguerre polynomials $\{L_0(x), L_1(x), L_2(x), \ldots\}$ can be constructed using the recurrence relation

$$(k+1)L_{k+1}(x) = (2k + 1 - x)L_k(x) - kL_{k-1}(x)$$

where $L_0 = 1$ and $L_1 = 1 - x$. These polynomials are known to be orthogonal on $(0, \infty)$ with respect to weight function e^{-x}.

(a) Construct $L_2(x)$. [2]

(b) Use L_0, L_1, L_2 to determine a polynomial $p_2(x)$ of degree at most 2 that approximates e^{-3x} on $(0, \infty)$ in the least squares sense. [18]

B5. Use a single step of the Runge-Kutta method of order 4 to solve the Initial Value problem

$$x'' - 3x' + 2x = 6e^{-t}, \; 0 \le x \le 1, \; x(0) = x'(0) = 2,$$

for both $x(0.1)$ and $x'(0.1)$. [20]

B6. (a) Derive the Leapfrog scheme

$$U_j^{n+1} = U_j^{n-1} - \alpha(U_{j+1}^n - U_{j-1}^n), \quad (4)$$

where $\alpha = a\frac{\Delta t}{\Delta x}$, for approximating advection equation (3). [5]

(b) Show that (4) is convergent provided $|\alpha| \leq 1$. [10]

(c) Determine the local truncation error for the Leapfrog scheme. [5]

B7. (a) Find a linear function $p_1(x)$ that approximates $\ln(x-2)$ on the closed interval $[3, 4]$ in the least squares sense. [10]

(b) (i) Use one step of the modified Euler method to evaluate the integral $\int_0^t e^{-\tau^2} d\tau$ when $t = 0.05$. [6]

(ii) Use two steps of Euler's method to solve the initial value problem

$$y'(x) = -y + x\sqrt{y},\ 2 \leq x \leq 4,\ y(2) = 2$$

for $y(2.1)$. [4]

B8. (a) Find a linear function $p_1(x)$ that best fits the data

i	0	1	2	3
x_i	-1	-2	-3	-4
y_i	9.4	2.5	3.7	1.6

in the least squares sense. [10]

(b) Show that the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{1}{2}\left[\frac{U_j^{n+1} - 2U_j^{n+1} + U_j^{n+1}}{h^2} + \frac{U_j^n - 2U_j^n + U_j^n}{h^2}\right]$$

for approximating the diffusion equation

$$u_t = u_{xx}$$

is unconditionally stable. [10]

END OF QUESTION PAPER