134

UNIVERSITY OF SWAZILAND

M 411

SUPPLEMENTARY EXAMINATION 2013/2014

BSc. /BEd. /B.A.S.S II

TITLE OF PAPER	:	Numerical Analysis II
COURSE NUMBER	:	M 411
TIME ALLOWED	:	3 HOURS
SPECIAL REQUIREMENTS	:	NONE. **NOT EVEN GRAPH IS REQUIRED.**

Instructions

(a) Candidates may attempt:

(i) ALL questions in Section A and

(ii) At most THREE questions in Section B.

(b) Each question should start on a fresh page.

(c) **THERE ARE NO SPECIAL REQUIREMENTS FOR THIS PAPER.**

135

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A3.

A1. Suppose that $w(x)$ is a weight function on $[a, b]$, and that

$$\int_a^b w = 4, \int_a^b wx = 3$$
$$\int_a^b wx^2 = 2, \int_a^b wx^3 = 1$$

Take $\phi_0(x) = 1$.

(a) Determine polynomials $\phi_1(x)$ and $\phi_2(x)$ of degrees 1 and 2 respectively, so that $S := \{\phi_0, \phi_1, \phi_2\}$ is an orthogonal set on $[a, b]$ with respect to w. [10]

(b) What does it mean to say that S is linearly independent on $[a, b]$? [2]

(c) Is S linearly independent on $[a, b]$? Justify your answer. [2]

A2. An initial value problem consists of finding $y(t)$ satisfying

$$\frac{dy}{dt} = e^{t-y}, \ 0 \le t \le 1, \ y(0) = 1 \tag{1}$$

(a) Use one step of the Runge-Kutta method of order 4 to approximate $y(0.5)$. [4]

(b) Let $f(t, y) = e^{t-y}$ and let $w_i \approx y(ih)$. A multi-step method for solving (1) with step size h is defined by

$$w_{i+2} = 4w_{i+1} - 3w_i - 2hf(t_i, w_i)$$

for each $i = 0, 1, \ldots, N-2$ where starting values w_0, w_1 are given.
Does this method converge? Justify your answer. [10]

A3. (a) Let Ω be the square whose vertices are the points $(0,0), (2,0), (2,2), (0,2)$ in $\mathbb{R}^2$, and let Γ be boundary of this rectangle. A boundary value problem consists of finding $u(x, y)$ satisfying

$$\nabla^2 u = 0 \text{ on } \Omega,$$
$$u = x + y \text{ on } \Gamma$$

Use the *five point formula* with a uniform grid on Ω to approximate $u(1, 1)$.
Be careful to explain any notation used. [5]

(b) Let a be a constant, let $j = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$. The Leapfrog scheme

$$U_j^{n+1} = U_j^{n-1} - \alpha(U_{j+1}^n - U_{j-1}^n), \tag{2}$$

where $\alpha = a\frac{\Delta t}{\Delta x}$, can be used for approximating the advection equation

$$u_t + au_x = 0 \tag{3}$$

(i) Derive equation (2). [5]

(ii) Specify the range of values of α for which the leapfrog scheme converges. Justify your answer. [2]

SECTION B (60 marks)

Candidates may attempt THREE questions being careful to number them B4 to B8.

B4. Legendre polynomials $\{P_0(x), P_1(x), P_2(x), \ldots\}$ can be constructed using the recurrence relation

$$(k+1)P_{k+1}(x) = (2k+1)xP_k(x) - kP_{k-1}(x)$$

where $P_0 = 1$ and $P_1 = x$. These polynomials are known to be orthogonal on $[-1, 1]$ with respect to weight function 1.

(a) Construct $L_2(x)$. [2]

(b) Use P_0, P_1, P_2 to determine a polynomial $p_2(x)$ of degree at most 2 that approximates $\sin 2\pi x$ on $[-1, 1]$ in the least squares sense. [18]

B5. Use two steps of the Runge-Kutta method of order 2 to solve the Initial Value problem

$$x'' - 3x' + 2x = 6e^{-t},\ 0 \le x \le 1,\ x(0) = x'(0) = 2,$$

for both $x(0.1)$ and $x'(0.1)$. [20]

B6. (a) Derive the Upstream scheme

$$U_j^{n+1} = U_j^n - \alpha(U_j^n - U_{j-1}^n), \tag{4}$$

where $\alpha = a\frac{\Delta t}{\Delta x}$, for approximating advection equation (3). [5]

(b) If $a > 0$, show that (2) is convergent provided $0 < \alpha \le 1$. [10]

(c) Determine the local truncation error for the Leapfrog scheme. [5]

B7. (a) Find a linear function $p_1(x)$ that approximates xe^x on the closed interval $[0, 1]$ in the least squares sense. [10]

(b) (i) Use two steps of the Euler method to evaluate the integral $\int_0^t e^{-\tau^3} d\tau$ when $t = 0.1$. [6]

(ii) Use one step of the Runge-Kutta method of order 4 to solve the initial value problem

$$y'(x) = xe^{3x} - 2y(x),\ 0 \leq x \leq 1,\ y(0) = 0$$

for $y(0.1)$. [4]

B8. (a) Let $T_0, T_1, T_2, \ldots$ denote the Chebyshev polynomials of the first kind.

(i) Show that if m and n are positive integers and $m > n$, then

$$T_m(x)T_n(x) = \frac{1}{2}[T_{m+n}(x) + T_{m-n}(x)]$$

[6]

(ii) Prove that $T_n(T_m(x)) = T_{nm}(x)$. [4]

(b) Use numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}$$

with $k = 1/27$ and $h = 1/3$ to solve differential equation $u_t = u_{xx}$ for $u(1/3, 1/27)$ and $u(2/3, 1/27)$ subject to boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,0) = x(1-x),\ 0 \leq x \leq 1.$$

[10]

END OF QUESTION PAPER