University of Swaziland

158

Final Examination, December 2013

B.A.S.S. , B.Sc, B.Eng, B.Ed

Title of Paper : Partial Differential Equations

Course Number : M415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Non programmable calculators may be used (unless otherwise stated).
5. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

1.1. Define a boundary condition and give three different types of boundary conditions. [4]

1.2. Determine the region in which the given partial differential equations are hyperbolic, parabolic or elliptic. $(u = u(x, y))$.

(a) $y^2 u_{xx} - 2u_{xy} + u_{yy} = u_x + 6y$. [4]

(b) $\sin^2 x u_{xx} - \sin(2x) u_{xy} + \cos^2 x u_{yy} = 1$. [3]

1.3. Solve the following partial differential equations, $(u = u(x, y))$.

(a) $u_{xx} + u = 0$. [4]

(b) $u_{xy} = 0$. [3]

(c) $xu_x - yu_y = 1 - xyu$. [5]

1.4. The cartesian coordinates $(x, y) \in \mathbb{R}^2$ and polar coordinates $(r, \theta) \in [0, \infty) \times [0, 2\pi]$ in the plane are related by $x = r\cos\theta$ and $y = r\sin\theta$. Transform the equation

$$xu_y - yu_x = 0$$

to polar coordinates and find the general solution. [11]

1.5. Find the partial differential satisfied by $u(x, y) = f(x)g(y)$. [6]

SECTION B: ANSWER ANY 3 QUESTIONS

2. Use Green's theorem

$$\iint\limits_{\Omega} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \oint\limits_{\Gamma} M dx + N dy$$

to show that the solution for the following partial differential equation

$$\begin{aligned} & u_{tt} - c^2 u_{xx} = 0, \qquad -\infty < x < +\infty, \quad t > 0 \\ & u(x,0) = f(x) \\ & u_t(x,0) = g(x) \end{aligned}$$

is given by

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c} \int\limits_{x-ct}^{x+ct} g(s) ds.$$

(Ω is the characteristic triangle and Γ is the boundary of the characteristic triangle).
Hence find the solution for the following partial differential equation

$$\begin{aligned} & u_{tt} - u_{xx} = 0, \qquad -\infty < x < +\infty, \quad t > 0 \\ & u(x,0) = x \\ & u_t(x,0) = e^x \end{aligned}$$

[20]

3. Reduce to canonical form

$$u_{xx} + u_{xy} - 2u_{yy} + 1 = 0,$$

and hence find the solution which passes through the curve

$$\Gamma: \quad u = u_y = x \text{ on } y = 0.$$

[20]

4. Consider the function

$$f(x) = \begin{cases} -1, & -\pi \le x < 0; \\ 0, & x = 0; \\ +1, & 0 < x \le \pi. \end{cases} \qquad f(x + 2\pi) = f(x).$$

(a) Find the fourier series expansion. [12]

(b) Use Parseval's identity to find the value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

[8]

5. Solve the following partial differential equations using the method of Laplace transforms

   (a) $u_{xt} + \sin t = 0, \quad u(x,0) = x,\ u(0,t) = 0.$ [10]

   (b) $xu_x + u_t = xt, \quad u(x,0) = 0,\ u(0,t) = 0.$ [10]

6. Show that the initial value problem with non-homogeneous boundary conditions

$$\begin{aligned} &u_t - u_{xx} = 0, \quad 0 < x < L, \quad t > 0 \\ &u(0,t) = T_1, \quad t \geq 0 \\ &u(L,t) = T_2, \quad t \geq 0 \\ &u(x,0) = f(x), \quad 0 \leq x \leq 1 \end{aligned}$$

can be transformed to an initial value problem with homogeneous boundary conditions. Hence find the general solution of initial value problem using separation of variables. [20]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{d^n f}{dt^n}(t)$	$s^n F(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$