University of Swaziland

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Supplementary Examination, July 2014

B.A.S.S. , B.Sc, B.Eng, B.Ed

Title of Paper : Partial Differential Equations

Course Number : M415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
   Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
   Answer ANY THREE questions.
   Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Non programmable calculators may be used (unless otherwise stated).
5. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

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1.1. (a) By eliminating the arbitrary function, find the partial differential equation satisfied by the following functions

(i) $u(x,y) = f\left(\frac{xy}{u}\right)$ [4]

(ii) $u = f(ax+by) + g(cx+dy)$
where $a,\ b,\ c$ and d are nonzero constants [6]

(b) Find the general solution of the partial differential

$$(au - by)u_x + (bx - cu)u_y = cy - ax.$$

where $a,\ b,\ c$ and d are nonzero constants. [10]

1.2. Determine the region in which the given partial differential equation is hyperbolic, parabolic or elliptic.

(a) $u_{xx} + y^2 u_{yy} = y.$ [4]

(b) $u_{xx} - yu_{xy} + xu_{yy} + u = 0.$ [4]

1.3. If $z = f(x,y),\ \ x = \frac{1}{2}(u^2 - v^2)$ and $y = uv$. Show that

$$u\frac{\partial z}{\partial v} - v\frac{\partial z}{\partial u} = 2\left(x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}\right).$$

[5]

1.4. Reduce the following partial differential equation to it's canonical form

$$u_{xy} + u_x + u_y = 3x.$$

[3]

1.5. Find the characteristics of the following partial differential equation

$$u_{xx} + 2u_{yy} + u_{xy} = 1$$

[4]

SECTION B: ANSWER ANY 3 QUESTIONS

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2. (a) Find the particular solution of the partial differential equation

$$x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u$$

that passes through the curve

$$\Gamma: \quad u = 0 \text{ on } x+y=0$$

[10]

(b) Given the function $u = u(x,y)$ and transformation $x = r\cos\theta,\ y = r\sin\theta$ write

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

in terms of r and θ [10]

3. Reduce the following equation

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0, \quad x \neq 0, \quad y \neq 0$$

to canonical form and then find the general solution [20]

4. Consider the function

$$f(x) = \begin{cases} \pi + x, & -\pi \leq x \leq 0; \\ \pi - x, & 0 \leq x \leq \pi. \end{cases}, \qquad f(x+2\pi) = f(x)$$

(a) Find the fourier series expansion for $f(x)$. [10]

(b) Use Parseval's identity to find the value of the sum

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

[10]

5. Solve the following boundary value problem

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H \\ &u(x,0) = 0, \quad 0 \leq x \leq L \\ &u(x,H) = 0, \quad 0 \leq x \leq L \\ &u(L,y) = 0, \quad 0 \leq y \leq H \\ &u(0,y) = f(y), \quad 0 \leq y \leq H \end{aligned}$$

where $f(y)$ is a defined function. [20]

6. Solve the following partial differential equation using the method of Laplace transforms

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\begin{aligned}
&u_{tt} = c^2 u_{xx} + \sin(\pi x), \quad 0 < x < 1, \quad t > 0 \\
&u(x,0) = 0, \quad 0 \le x \le 1 \\
&u_t(x,0) = 0, \quad 0 \le x \le 1 \\
&u(0,t) = 0, \quad t \ge 0 \\
&u(1,t) = 0, \quad t \ge 0.
\end{aligned}
$$

[20]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{d^n f}{dt^n}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$