UNIVERSITY OF SWAZILAND

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FINAL EXAMINATION, 2013/2014

B.Sc. IV, BASS IV, BED. IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked.**
2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

A1. (a) Define

(i) An ideal N in a ring R
(ii) A divisor of zero in a ring R
(iii) The characteristic of a ring R
(iv) Unity in a ring R
(v) A unit in a ring R (10)

(b) Prove that in the ring $\mathbb{Z}_n$
(i) the divisor of zero are those elements that are not relatively prime to n.
(ii) the elements that are relatively prime to n cannot be divisers of zero (10)

A2. (a) Give a definition of the following
(i) an intergral domain
(ii) a field (6)
(b) Prove that a finite integral domain is a field. (10)
(c) Give an example of a integral domain that is not a field. (4)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

(a) Show that if D is an integral domain then the ring $D[x]$ of polynomials is also an integral domain. (7)

(b) The polynomial $x^4+2x^3+x^2+x+1$ has a linear factor in $\mathbb{Z}_3[x]$. Find the factorization into irreducible polynomials in
$\mathbb{Z}_3[x]$. (7)

(c) Show that $x = \sqrt{1-\sqrt{2}}$ is algebraic over $\mathbb{Q}$. Find the minimum polynomial and the degee of α

(i) over $\mathbb{R}$
(ii) over $\mathbb{Q}$. (6)

QUESTION B4 [*20 Marks*]

(a) Show that the polynomial x^2+x+1 is irreducible in $\mathbb{Z}_2[x]$ (6)

(b) Let α be a zero of x^2+x+1 in the extension field of $\mathbb{Z}_2$ i.e, $E = \mathbb{Z}_2(\alpha)$.

(i) Write down all the elements of $\mathbb{Z}_2(\alpha)$ (6)
(ii) Construct the multiplication table for $\mathbb{Z}_2(\alpha)$, showing the inverses for each non zero element. (8)

QUESTION B5 [*20 Marks*]

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(a) Which of the following sets is a ring with the usual operations of addition and multiplication? In each case, either prove that it is a ring or explain why it is not

(i) the set $\{1, -1, i, -i\}$

(ii) The set $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$. (10)

(b) Show that the rings $\mathbb{Z}$ and $3\mathbb{Z}$ are NOT isomorphic. (5)

(c) Show that $\mathbb{Z}_n$ is a field $\Leftrightarrow$ n is prime (5)

QUESTION B6 [*20 Marks*]

(a) State Eisenstein's criterion for irreducibility [3]

(b) Use Eisenstein's criterion to show that $f(x) = 26x^5 - 5x^4 + 25x^2 - 10$ is irreducible over Q (4)

(c) Find all zeros of $x^3 + 2x + 2$ in $\mathbb{Z}_7$ (5)

(d) Find the quotient and the remainder when $f(x)$ is divided by $d(x)$ in $\mathbb{Z}_5[x]$if $f(x) = x^4 - 3x^3 + 2x^2 + 4x - 1$ and $d(x) = x^2 - 2x + 3$ (8)

QUESTION B7 [*20 Marks*]

(a) Prove that if R is a ring with unity and H is an ideal of R containing a unit, then $N = R$ (10)

(b) Let R be a commutative ring. For an arbitrary element n in R, form the set $N = \{nr : r \in R\}$
Prove that N is an ideal in R. [10]

END OF EXAMINATION PAPER