University of Swaziland

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Supplementary Examination, 2013/2014

B.Sc. IV, BASS IV, BED. IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked.**
2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

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A1. (a) (i) Define an ideal N of a ring R. (5)

(ii) Find all ideals N of $\mathbb{Z}_{12}$ and all maximal ideal of $\mathbb{Z}_{18}$ (5)

(b) (i) Prove that every finite integral domain is a field. (5)

(ii) Show that for a field F, the set of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \text{ for } a, b \in F$$

is a right ideal but not a left ideal of $M_2(F)$. (5)

A2. (a) In a ring $\mathbb{Z}_n$ show that

(i) divisor c of zero are those elements that are NOT relatively prime to n. (5)

(ii) elements that are relatively prime can't be zero divisors (5)

(b) (i) Given an example of a ring R with unity 1 that has a subring $\mathbb{R}^1$ with unity 1^1, where $1 \neq 1^1$. (5)

(ii) Describe all units in the ring $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$ (5)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

(a) Let f be a polynomial over $\mathbb{Z}$ which is irreducible over $\mathbb{Z}$.
Show that f considered as a polynomial over $\mathbb{Q}$ is also irreducible. (10)

(b) Classify each of the given $\alpha \in \mathbb{C}$ as algebraic or transcendantal over the given field F.
If α is algebraic over F, find $\deg(\alpha, F)$

(i) $\alpha = 1 + i, \quad F = \mathbb{Q}$

(ii) $\alpha = \sqrt{\pi}, \quad F = \mathbb{Q}(\pi)$

(iii) $\alpha = \pi^2, \quad F = \mathbb{Q}$

(iv) $\alpha = \pi^2, F = \mathbb{Q}(\pi)$

(v) $\alpha = \pi, \quad F = \mathbb{Q}(\pi^3)$ (10)

QUESTION B4 [*20 Marks*]

(a) Prove that if D is an integral domain, then $D[x]$ is also an integral domain. (10)

(b) Decide the irreducibility or otherwise of

(i) $x^3 - 7x^2 + 3x + 3 \in Q[x]$ (5)

(ii) $2x^{10} - 25x^3 + 10x^2 - 30 \in \mathbb{Q}[x]$ (5)

QUESTION B5 [*20 Marks*]

(a) (i) Show that the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ is NOT a field. (5)

(ii) Find a polynomial of degree > 0 in $\mathbb{Z}_4[x]$ that is a unit. (5)

(b) (i) Show that $(a+b)(a-b) = a^2 - b^2$ for all a and b in a ring R, f if and only if R is commutative (5)

(ii) Show that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are NOT isomorphic. (5)

QUESTION B6 [*20 Marks*]

(a) Suppose F is a field, f_n is an irreducible polynomial over F and g, h are polynomials over F such that f divides gh. Show that either f divides g or f divides h. [10]

(b) Let $\varphi_\alpha : \mathbb{Z}_7[x] \rightarrow Z_7$. Evaluate each of the following for the indicated evaluation homomorphism.

(i) $\varphi_5[(x^3+2)(4x^2+3)(x^7+3x^2+1)]$ (5)

(ii) $\varphi_4[3x^{106} + 5x^{99} + 2x^{53}]$ (5)

QUESTION B7 [*20 Marks*]

(a) Determine whether each of the following polynomials in $\mathbb{Z}[x]$ satifies an Eissenstein criteria for irreducibility

(i) $8x^3 + 6x^2 - 9x + 24$ (5)

(ii) $2x^{10} - 25x^3 + 10x^2 - 30$ (5)

(b) Let α be a zero of $x^2 + 1$ in an extension field of $\mathbb{Z}_3$. Give the multiplication and addition tables for the nine elements of $\mathbb{Z}_3(\alpha)$. (10)

END OF EXAMINATION PAPER