UNIVERSITY OF SWAZILAND

110

SUPPLEMENTARY EXAMINATIONS 2013/2014

B.Sc. / B.Ed. / B.A.S.S. IV

TITLE OF PAPER	:	METRIC SPACES
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ALL QUESTIONS IN SECTION A. 3. ANSWER ANY THREE QUESTIONS IN SECTION B.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

SECTION A

IL1

QUESTION 1

Let (X, d) be a metric space. Define the following:

(a) the distance from $x \in X$ to a subset $A \subset X$; [2]

(b) the diameter of $A \subset X$; [2]

(c) the distance between two subsets, A and B, of X; [2]

(d) a bounded subset $A \subset X$; [2]

(e) a bounded mapping g from a nonempty set Y to X; [2]

(f) a convergent sequence (x_n) in X; [2]

(g) a Cauchy sequence (x_n) in X; [2]

(h) a subspace (Y, d_Y) of (X, d); [2]

(i) an open ball $B(a, r)$ in (X, d); [2]

(j) an open subset F of X. [2]

QUESTION 2

162

(a) Can you find a metric space (X, d) where:

(i) The interval $[0, 1]$ is both open and closed? [2]

(ii) The interval $[0, \frac{1}{2}]$ is open but not closed? [2]

Justify your answer in each case.

(b) Describe open balls $B(a, 3)$, where $a = (2, 3)$ in $\mathbb{R}^2$ with respect to the following metrics:

(i) the Chicago metric; [3]

(ii) the London (or UK)-rail metric; [4]

(iii) the New York metric; [4]

(iv) the Raspberry pickers' metric. [3]

(c) Give an example of a metric space in $\mathbb{R}$, equipped with the usual metric, such that $\text{diam}(A^{\circ}) < \text{diam}(A)$. [2]

SECTION B

QUESTION 1

Let $A \subset \mathbb{R}^2$ be the region bounded by the unit disc centered at the origin. Find diam(A) with each of the following metrics:

(a) the Max metric; [4]

(b) the Chicago metric; [5]

(c) the London (or UK)-rail metric; [3]

(c) the New York metric; [4]

(d) the Raspberry pickers' metric. [4]

QUESTION 2

(a) Let (X, d) be a metric space, and let (x_n) and (y_n) be two sequences in X such that (y_n) is a Cauchy sequence and $d(x_n, y_n) \to 0$ as $n \to \infty$.Prove that:

- (i) (x_n) is a Cauchy sequence in X; [5]
- (ii) (x_n) converges to a limit x in X if and only if (y_n) also converges to x in X. [5]

(b) Prove that every Cauchy sequence in a metric space (X, d) is bounded. [4]

(c) Let (X, d) be a metric space, and let d' be the metric on X defined by

$$d'(x, y) = \min\{1, d(x, y)\}.$$

Prove that (x_n) is a Cauchy sequence in (X, d) if and only if (x_n) is a Cauchy sequence in (X, d'). [6]

QUESTION 3

164

(a) If a sequence (x_n) is convergent and has limit x, prove that every subsequence $(x_{n_k})_{k\geq 1}$ of (x_n) is convergent and has the same limit x. [4]

(b) Let $X = C[0,1]$, the set of all continuous functions on $[0,1]$, and let d be the metric on X defined by

$$d(f,g) = \int_0^1 |f(x) - g(x)|\,\mathrm{d}x.$$

For each $n \in \mathbb{N}$, define f_n by $f_n(x) = x^n$ for all $x \in [0,1]$.

(i) Show that the sequence (f_n) converges in X, and find its limit f. [3]

(ii) Show that the function f in Part (i) is not the pointwise limit of the sequence (f_n). [3]

(c) Let d be the metric on $X = C[a,b]$ defined by

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

Let (f_n) be a sequence in $C[a,b]$, and suppose that (f_n) converges uniformly on $[a,b]$ to some function f.

(i) Prove that f is continuous on $[a,b]$, and hence show that (f_n) converges in (X,d). [5]

(ii) Prove that $\int_a^b f_n(x)\,\mathrm{d}x \longrightarrow \int_a^b f(x)\,\mathrm{d}x$ as $n \to \infty$. [5]

QUESTION 4

(a) Given a function $f : (X, d_1) \longrightarrow (X, d_2)$,

(i) When is f said to be continuous in the $\varepsilon - \delta$ sense?

(ii) Give an equivalent definition in terms of open sets.

(iii) Assuming f is continuous at x_0, prove that

$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0).$$

[14]

(b) Prove that the function $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\pi(x, y) = x$ is continuous when $\mathbb{R}^2$ and $\mathbb{R}$ are equipped with their usual metrics. Is π uniformly continuous? Justify your answer. [6]

QUESTION 5

(a) When are two subsets A and B of a metric space said to be separated? [2]

(b) Verify that two nonempty disjoint closed sets in a metric space are separated. [2]

(c) Give two alternate definitions of connectedness of a subset M of a metric space X. [4]

(d) (i) Prove that if X is a connected metric space and $f : X \longrightarrow \mathbb{R}$ is a continuous function, then $f(X)$ is connected.

(ii) Deduce that if $f : [0, 1] \longrightarrow [0, 1]$ is continuous, then there exists an $x \in [0, 1]$ such that $f(x) = x$. [12]

END OF EXAMINATION