UNIVERSITY OF SWAZILAND

FINAL EXAMINATION, 2015/2016

B.Sc. II, B.Eng II, BASS II, BED. II

Title of Paper : Linear Algebra

Course Number : M220

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked.**
2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

A1. (a) Let V be all ordered pairs of real numbers. Define addition and scalar multiplication as follows; $(x, y) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and $\alpha(x, y) = (\alpha x_1 + \alpha - 1, \alpha y_1 + \alpha - 1)$. Show that V is a vector space. (12)

(b) Show that the vectors $v_1 = (4, 3, -2, 3), v_2 = (6, 5, -5, 1)$ and $v_3 = (2, -1, 3, 5)$ are linearly dependent and express one as a linear combination of the other two. (5)

(c) Determine whether the following has a non-trivial solution:

$$\begin{aligned} 2x + y - z + 2w &= 0 \\ x + y + z + w &= 0 \\ 3x + 2y + 2z + 2w &= 0 \end{aligned}$$

(3)

A2. (a) Use Crammer's rule to solve (i) and use Gaussian elimination to solve (ii).

(i) $$\begin{pmatrix} 1 & -3 & 1 \\ -2 & 2 & -1 \\ 4 & -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

(ii) $$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$ (10)

(b) Find the inverses by inspection.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(i) State the Cayley-Hamilton Theorem. (4)

(ii) Given that $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$, verify Cayley-Hamilton theorem for the matrix A. (6)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

(a) Determine whether the sets of vectors in the vector space V are linearly dependent or independent

(i) $\{2x^2 + x, x^2 + 3, x\} \quad V = P_2(x)$.

(ii) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} V = M_2(\mathbb{R})$ (10)

(b) Let $S = \{v_1, v_2, \cdots, v_n\}$ be a set of non-zero vectors in a vector space V. Prove that S is linearly dependent $\Leftrightarrow$ one of the vectors is a linear combination of the preceeding vectors in S. (10)

QUESTION B4 [*20 Marks*]

(a) Find conditions for λ and μ for which the following system has

(i) a unique solution

(ii) no solution or

(iii) infinitely many solutions.

$$\begin{aligned} x + y - 4z &= 0 \\ 2x + 3y + z &= 1 \\ 4x + 4y + \lambda z &= \mu \end{aligned}$$

(10)

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(x, y) = (x - 2y, 2x + y, x + y)$

(i) Find the standard matrix of T

(ii) Find the matrix of T with respect to β^1 and β where $\beta^1 = \{(1, -1), (0, 1)\}$ and $\beta = \{(1, 1, 0)(0, 1, 1), (1, -1, 1)\}$ (10)

QUESTION B5 [*20 Marks*]

(a) Given that $A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 3 & -2 \end{pmatrix}$ Use the augemented matrix $[A : I]$ to find A^{-1} (10)

(b) (i) In (a) above find a finite sequence of elementary matrices $E_1, \cdots, E_k$ such that $E_k E_{k-1}, \cdots E_1 A = I$.

(ii) Show that $A^{-1} = E_k E_{k-1} \cdots E_1$

(c) Evaluate the following determinant using cofactor expansion along the second row

$$\begin{vmatrix} 3 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -3 & 2 \end{vmatrix}$$ (4)

(d) Determine whether the system has a nontrivial solution

$$\begin{aligned} -x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \\ 3x_1 + x_2 - x_3 + 2x_4 &= 0 \\ x_1 - 2x_2 + 3x_3 - x_4 &= 0 \end{aligned}$$

(3)

QUESTION B6 [*20 Marks*]

(a) Prove that if a homogeneous system has more unknowns than the number of equations then it has a non-trivial solution. (10)

(b) Find the characteristic polynomial eigenvalues and eigenvectors of the following matrix $\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$ (10)

QUESTION B7 [*20 Marks*]

(a) Show that B is a basis of $\mathbb{R}^3$ where $B = \{(0,2,1),(1,0,2),(1,-1,0)\}$ (5)

(b) Show that the vector $\begin{pmatrix} 12 \\ 12 \\ -3 \end{pmatrix}$ is a linear combination of the vectors $(2,0,1)^T, (4,2,0)^T, (1,3,-1)^T$ (5)

(c) Let $B_1 = \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ and $B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ be bases. Find the transition matrix from B_1 to B_2 (10)

END OF EXAMINATION PAPER