UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2015/2016

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ALL QUESTIONS IN SECTION A. 3. ANSWER ANY THREE QUESTIONS IN SECTION B.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

SECTION A

QUESTION 1

(a) What do you understand by the following?

(i) Universal Instantiation; [3]

(ii) Universal Generalization; [3]

(iii) Existential Instantiation; [3]

(iv) Existential Generalization. [3]

(b) State the Pigeonhole Principle. [2]

(c) State the Principle of Mathematical Induction II. [2]

(c) State the Principle of Strong Mathematical Induction. [2]

(d) State the Fundamental Theorem of Arithmetic. [2]

QUESTION 2

(a) (i) Suppose you want to show that $A \implies B$ is **false**, where A and B are statements. How should you do this? What should you try to show about the truth of A and B?. [2]

(ii) Apply your answer of part (i) to show that the statement "If x is a real number that satisfies $-3x^2 + 2x + 8 = 0$, then $x > 0$" is false. [3]

(b) For each of the following, write the *converse* and the *contrapositive*:

(i) If n is an integer for which n^2 is even, then n is even. [3]

(ii) Suppose that t is an angle between 0 and π. If t satisfies $\sin(t) = \cos(t)$, then $t = \frac{\pi}{4}$. [3]

(c) Write the negation of the following definition:

The function f of one variable is a *convex function* if and only if for all real numbers x and y and for all real numbers t with $0 \leq t \leq 1$, it follows that $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.

[9]

SECTION B

QUESTION 3

(a) Prove that if $A \Rightarrow B$, $B \Rightarrow C$, and $C \Rightarrow A$, then A is equivalent to B and A is equivalent to C. [11]

(b) Determine the following sets:

(i) $\{m \in \mathbb{N} : \exists n \in \mathbb{N} \text{ with } m \leq n\}$; [2]

(ii) $\{m \in \mathbb{N} : \forall n \in \mathbb{N} \text{ we have } m \leq n\}$. [2]

(c) Let a be an algebraic number and let r be a rational number. Show that ra is an algebraic number. [5]

QUESTION 4

(a) Prove that between any two *distinct irrational* numbers, there is a rational number and an irrational number. [10]

(b) Define the following:

(i) Fallacy of affirming the conclusion; [2]

(ii) Fallacy of denying the antecedent. [2]

(c) Using truth tables, analyze the following argument and state whether it is valid or invalid

"All Germans are Europeans.
My neighbor is not a German.
Therefore my neighbor is not a European." [6]

QUESTION 5

(a) Describe a modified induction procedure that could be used to prove statements of the form:

(i) For all integers $n \leq k$, $P(n)$ is true, where $P(n)$ is a statement containing the integer n. [3]

(ii) For all integers n, $P(n)$, where $P(n)$ is as in Part (i). [4]

(iii) For every positive odd integer, something happens. [3]

(b) For all non-negative integers n define the number u_n inductively as

$$\begin{aligned} u_0 &= 0, \\ u_{k+1} &= 3u_k + 3^k \qquad \text{for } k \geq 0. \end{aligned}$$

Prove that $u_n = n3^{n-1}$ for all non-negative integers n. [4]

(c) If $f(n) = 3^{2n} + 7$, where n is a natural number, show that $f(n+1) - f(n)$ is divisible by 8. Hence prove by induction that $3^{2n} + 7$ is divisible by 8. [6]

QUESTION 6

(a) Let p_1 and p_2 be distinct prime numbers. Prove that the real numbers $\sqrt{p_1}+\sqrt{p_2}$ and $\sqrt{p_1} - \sqrt{p_2}$ are irrational. [10]

(b) Prove that the square root of a natural number is rational if and only if the natural number is a perfect square. [10]

QUESTION 7

(a) (i) Define an equivalence relation. [2]

(ii) Show that the relation

$$\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod{2}\}$$

is an equivalence relation. What are the equivalence classes of $\mathcal{R}$? [12]

(b) (i) Define the composition $f \circ g$ of any two functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$. [2]

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions defined by $f(x) = \sin x$ and $g(x) = x^2 + 2$ for all $x \in \mathbb{R}$. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$. [4]

END OF EXAMINATION