University of Swaziland

Final Examination – May 2016

BSc III, BEd III, BEng III, BASS III

Title of Paper : Vector Analysis
Course Number : M312
Time Allowed : Three (3) hours

Instructions:

1. This paper consists of 2 sections.
2. Answer ALL questions in Section A.
3. Answer ANY THREE (3) questions in Section B.
4. Show all your working.

Section A

Answer ALL Questions in this section

A.1 a. Evaluate

i. $\displaystyle\int_0^1 (\ln x)^6 \mathrm{d}x$ [4 marks]

ii. $\displaystyle\int_0^{2\pi} \cos^6 \theta \mathrm{d}\theta$ [4 marks]

b. Using the Rodrigue's formula $L_n(x) = \dfrac{e^x}{n!}\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n e^{-x}\right)^n$, find the Laguerre polynomial $L_2(x)$. [4 marks]

A.2 For the vector function

$$\boldsymbol{F} = \boldsymbol{i}x\sin y + \boldsymbol{j}\left(z^2 + e^{-xy}\right) + \boldsymbol{k}\left(3x^2y - 2\right),$$

find

a. $\dfrac{\partial^2 \boldsymbol{F}}{\partial x \partial y}$ [4 marks]

b. $\nabla \cdot \boldsymbol{F}$ [3 marks]

c. $\nabla \times \boldsymbol{F}$. [5 marks]

A.3 Given that

$$\frac{\mathrm{d}^2\boldsymbol{F}}{\mathrm{d}t^2} = 6t\boldsymbol{i} - 24e^{-2t}\boldsymbol{j} + 4\sin 2t\boldsymbol{k},$$

and $\boldsymbol{F}(0) = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$, $\dfrac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}(0) = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, find the vector function $\boldsymbol{F}(t)$.

[8 marks]

A.4 Find the work done in moving a particle once around the circle $x^2 + y^2 = 16$ in the xy-plane under the force field

$$\boldsymbol{F} = (2x + 3y)\boldsymbol{i} + 3x\boldsymbol{j} - (x^2e^x - y^6)\boldsymbol{k}.$$ [8 marks]

Section B

Answer ANY THREE (3) Questions in this section

B.1 a. The formula

$$\int_{-\infty}^{\infty} e^{-ax^2}\mathrm{d}x = \sqrt{\frac{\pi}{a}}$$

arises in the analysis of the Gaussian distribution, applicable in numerous areas of study including probability theory, quantum field theory, image processing and artificial neural networks. Use Gamma functions to prove this formula. [6 marks]

b. Use the formula $\int_{-1}^{1} f(x)P_n(x)\mathrm{d}x = \frac{1}{2^n n!}\int_{-1}^{1}\left(1-x^2\right)^n \frac{\mathrm{d}^n f}{\mathrm{d}x^n}\mathrm{d}x$ to prove that[1]

$$\int_{-1}^{1} x^n P_n(x)\mathrm{d}x = \frac{2^{n+1}(n!)^2}{(2n+1)!},$$ [14 marks]

where $P_n(x)$ is the Legendre polynomial of degree $n \in \mathbb{Z}^+$.

B.2 a. Given the vector function $\boldsymbol{F}(x,y,z) = \boldsymbol{i}F_1(x,y,z) + \boldsymbol{j}F_2(x,y,z) + \boldsymbol{k}F_3(x,y,z)$, where F_1, F_2 and F_3 are differentiable, prove that

$$\nabla\cdot(\nabla\times\boldsymbol{F}) = 0.$$ [8 marks]

b. Consider the vector field

$$\boldsymbol{F} = (y^2+2)\boldsymbol{i} + (2xy - z^2)\boldsymbol{j} - (2yz+10)\boldsymbol{k}.$$

i. Show that $\boldsymbol{F}$ is a conservative force field [4 marks]

ii. Find the scalar potential Φ such that $\boldsymbol{F} = \nabla\Phi$. [5 marks]

iii. Hence, or otherwise, find the work done in moving a particle from $(-1,2,2)$ to $(4,-3,8)$ in the force field $\boldsymbol{F}$. [3 marks]

B.3 a. Given the position vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and its magnitude $r = |\boldsymbol{r}| = \sqrt{x^2+y^2+z^2}$, where $n \in \mathbb{R}$, prove that

i. $\nabla\left(r^n\right) = nr^{n-2}\boldsymbol{r}$ [5 marks]

ii. $\nabla^2\left(r^n\right) = n(n+1)r^{n-2}$. [9 marks]

b. Find the volume of the region enclosed by the sphere $x^2+y^2+z^2 = a^2$ and the cone $z = \sqrt{x^2+y^2}$. [6 marks]

[1] Useful Formula: $\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2})$.

B.4 a. Find the work done in moving a particle in the force field

$$\boldsymbol{F} = 3x\boldsymbol{i} + (2z - y)\boldsymbol{j} + 2y\boldsymbol{k}$$

from $(2, 1, 1)$ to $(4, 4, 8)$ along the curve $x = 2t,\ y = t^2,\ z = t^3,\ 1 \leqslant t \leqslant 2$.

[7 marks]

b. Verify Stokes' theorem

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n} \mathrm{d}S$$

where $\boldsymbol{F} = (2x - 9y)\boldsymbol{i} + y^3 z^4 \boldsymbol{j} + y^4 z^3 \boldsymbol{k}$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

[13 marks]

B.5 a. Consider the formula

$$A = \tfrac{1}{2} \oint_C x\mathrm{d}y - y\mathrm{d}x$$

for the area of a region bounded by the simple closed curve C.

i. Show that in polar coordinates $(x = r\cos\theta,\ y = r\sin\theta)$, the formula becomes

$$A = \tfrac{1}{2} \int_0^{2\pi} r^2 \mathrm{d}\theta.$$

[6 marks]

ii. Hence, or otherwise, find the area of the region inside the dimpled limacon $r = 3 + 2\sin\theta$.

[4 marks]

b. Consider the cylindrical polar coordinate system defined by

$$x = \rho\cos\phi,\ \ y = \rho\sin\phi, z = z.$$

i. Derive the scale factors $h_\rho,\ h_\phi$ and h_z.

[4 marks]

ii. Hence, or otherwise, use the formulae

$$\nabla\Psi = \frac{1}{h_1}\frac{\partial\Psi}{\partial u_1}e_1 + \frac{1}{h_2}\frac{\partial\Psi}{\partial u_2}e_2 + \frac{1}{h_3}\frac{\partial\Psi}{\partial u_3}e_3$$

$$\nabla \cdot \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1) + \frac{\partial}{\partial u_2}(h_1 h_3 F_2) + \frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right]$$

to derive the formula for

$$\nabla^2\Psi$$

in cylindrical polar coordinates.

[6 marks]

END OF EXAMINATION