University of Swaziland

Supplementary Examination – July 2016

BSc III, BEd III, BEng III, BASS III

Title of Paper	:	Vector Analysis
Course Number	:	M312
Time Allowed	:	Three (3) hours

Instructions:

1. This paper consists of 2 sections.
2. Answer ALL questions in Section A.
3. Answer ANY THREE (3) questions in Section B.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A

Answer ALL Questions in this section

A.1 a. Evaluate

i. $\displaystyle\int_0^\infty t^4 e^{-4t}\mathrm{d}x$ [4 marks]

ii. $\displaystyle\int_0^1 x^{\frac{7}{2}}(1-x)^{\frac{9}{2}}\mathrm{d}x$ [4 marks]

b. Using the generating function $G(x,h) = \dfrac{1}{\sqrt{1-2xh+h^2}} = \displaystyle\sum_{n=0}^{\infty} h^n P_n(x)$ for Legendre polynomials, show that $P_n(1) = 1$ for all $n \in \mathbb{Z}^+$. [4 marks]

A.2 Consider the scalar function

$$\Psi = x^2y + y^2x - 2xy + z^2.$$

Find

a. $\nabla\Psi$ [4 marks]

b. $\nabla^2\Psi$ [3 marks]

c. $\nabla \times (\nabla\Psi)$. [5 marks]

A.3 Find the value of

a. $\displaystyle\int_0^4 \Big[(2-4t)\boldsymbol{i} + te^{-t}\boldsymbol{j} - \sin t\boldsymbol{k}\Big]\mathrm{d}t$ [5 marks]

b. $\displaystyle\iiint_V 45xyz\mathrm{d}V$ where V is the region bounded by the plane $2x+3y+4z = 12$ in the first octant. [5 marks]

c. $\displaystyle\int_C (x-1)\mathrm{d}x + (2x+y)\mathrm{d}y$ where C is the curve $x = 2\cos t,\ y = 2\sin t$ from $t = 0$ to $t = \frac{3\pi}{4}$. [6 marks]

Section B

Answer ANY THREE (3) Questions in this section

B.1 a. Show that the set of functions

$$\left\{1,\ 1-x,\ \tfrac{1}{2}\left(x^2-4x+2\right)\right\}$$

is orthogonal in the interval $[0,\infty)$ with respect to the weight function $w(x)=e^{-x}$. [10 marks]

b. Prove that

i. $\displaystyle\sum_{k=m}^{\infty} B(k,n)=B(m,n-1)$ [5 marks]

ii. $\displaystyle\int_0^1 x^{2m+1}(1-x^2)^n \mathrm{d}x=\frac{m!n!}{2(m+n+1)!}$ [5 marks]

B.2 a. If $\boldsymbol{v}=\boldsymbol{w}\times\boldsymbol{r}$, prove that

$$\boldsymbol{w}=\tfrac{1}{2}(\nabla\times\boldsymbol{v})$$

where $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ and $\boldsymbol{w}$ is a constant vector. [8 marks]

b. Consider the vector field

$$\boldsymbol{F}=(xz^2)\boldsymbol{i}-y\boldsymbol{j}+x^2z\boldsymbol{k}.$$

i. Show that $\boldsymbol{F}$ is a conservative force field [4 marks]

ii. Find the scalar potential ϕ such that $\boldsymbol{F}=\nabla\phi$. [5 marks]

iii. Hence, or otherwise, ind the work done in moving a particle from $(1,1,-2)$ to $(5,4,10)$ in the force field $\boldsymbol{F}$. [3 marks]

B.3 a. Let $f(r)$ be a differentiable function and $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$. Prove that

i. $\nabla f(r)=\dfrac{f'(r)\boldsymbol{r}}{r}$ [5 marks]

ii. $\nabla\times\Big(\boldsymbol{r}f(r)\Big)=0$ [9 marks]

b. Find the volume of the region enclosed by the plane $z=a$ and the paraboloid $z=x^2+y^2$. [6 marks]

B.4 a. Find the work done in moving a particle in the force field

$$\boldsymbol{F} = (2x+1)\boldsymbol{i} + (2y-x)\boldsymbol{j} + 2z\boldsymbol{k}$$

from $(0,0,0)$ to $(2,2,4)$ along the curve $x=t,\ y=t,\ z=2t,\quad 0 \leqslant t \leqslant 2.$

[7 marks]

b. Verify Stokes' theorem

$$\oint_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F})\cdot \boldsymbol{n}\mathrm{d}S$$

where $\boldsymbol{F} = (y,-z,x)$ and S is the part of the plane $x+3y+5z = 150$ in the first octant.

[13 marks]

B.5 Consider the cylindrical polar coordinate system defined by

$$x = \rho\cos\phi,\quad y = \rho\sin\phi, z = z.$$

a. Derive the scale factors $h_\rho,\ h_\phi$ and h_z. [4 marks]

b. Find the unit vectors $\boldsymbol{e}_\rho,\ \boldsymbol{e}_\phi$ and $\boldsymbol{e}_z$ in terms of $\boldsymbol{i},\ \boldsymbol{j}$ and $\boldsymbol{k}$. [4 marks]

c. Hence show that $\boldsymbol{e}_\rho,\ \boldsymbol{e}_\phi$ and $\boldsymbol{e}_z$ form an orthogonal coordinate system.

[2 marks]

d. Use the formula

$$\nabla\Psi = \frac{1}{h_1}\frac{\partial\Psi}{\partial u_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial\Psi}{\partial u_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial\Psi}{\partial u_3}\boldsymbol{e}_3$$

to derive the formula for

$$\nabla\Psi$$

in cylindrical polar coordinates. [4 marks]

d. Hence, or otherwise, show that

$$\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial y}\right)^2 = \left(\frac{\partial\Psi}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial\Psi}{\partial \phi}\right)^2.$$

[6 marks]

END OF EXAMINATION