University of Swaziland

Final Examination, 2015/2016

B.Sc. III, BASS III, B.Ed III

Title of Paper : Complex Analysis

Course Number : M313

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked.**
2. Show all your working.

Special Requirements: None

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: Answer ALL Questions

A1. (a) Represent a complex number in algebraic, trigonometric and exponential form.

(b) Show $Im(iz) = Rez$.

(c) Define and give example of an open set of complex numbers. (2,2,3

A2. (a) A semicircle $|z| = 1, \quad Rez > 0$, is transformed by a linear transformation into a semicircle $|w + 2| = 2$ with negative imaginary part. Find this linear transformation.

(b) Using Cauchy-Riemann equations (CRE) state the necessary conditions theorem for the function by differentiable at z_0. (5,5)

A3. Give definition and example of

(a) entire function

(b) harmonic function. (2,2)

A4. Prove that if $f(z) = u(x, y) + iv(x, y)$ is analytic in domain D then v is harmonic conjugate of u. (7)

A5. State the

(a) Cauchy integral formula for

$$\int_c \frac{f(z)dz}{z - z_0}$$

(b) Taylor series theorem (3,3)

A6. Classify and give examples of isolated points. (6)

SECTION B: Answer any *THREE* Questions

QUESTION B1 [*20 Marks*]

B1. (a) Construct the line $Re\frac{1}{z+2} = \frac{1}{4}$. (4)

(b) Find and sketch the region into which the square $x = 0, x = 1, y = 0, y = 1$ is mapped by transformation $w = e^z$. (4)

(c) Explain what is meant by the statement "The function $f(z)$ is continuous at z_0." (2)

(d) Find the limits. Give your reasonings

(i) $\lim\limits_{x \to -1} \frac{z - 3i}{2z + 2}$,

(ii) $\lim\limits_{x \to \infty} \frac{2z + i}{z + 1}$. (2,3)

(e) Using just the definition of derivative find f' if

(i) $f(z) = (z - 3)^2$,

(ii) $f(z) = lmz$. (2,3)

QUESTION B2 [*20 Marks*]

B2. (a) Using CRE

(i) State the sufficient conditions theorem for existence of $f'(z)$ and thus

(ii) Check if there is f' and g' for $f = z^2 + 3z + 3$ and $g = e^x e^{iv}$, respectively. Find f' and g'. (3,3)

(b) Derive CRE in polar coordinates. (6)

(c) Find the analytic function $f(z) = u(x, y) + iv(x, y)$, given that the imaginary part $v = 3x + 2xy$ and $f(-i) = 2$. (8)

QUESTION B3 [*20 Marks*]

B3. (a) Evaluate $\int_c \sqrt{z}dz$, where c is the upper half of the circle $|z| = 3$, from $z = 3$ to $|z = -3$. (6)

(b) Apply Cauchy integral formula to evaluate

(i) $\int_c \frac{dz}{z^2(z^2+25)}$, if

$c = \{z : |z| = 4 = 4$ in positive direction, and $|z| = 2$ in negative direction.$\}$

(ii) $\int_c \frac{dz}{z^2+2z}$, where c is a positively oriented circle $|z| = 1$. (3,5)

(c) State the Laurent series theorem. (3)

(d) Expand $f(z) = \frac{1}{1+z}$ in Maclaurin series. (4)

QUESTION B4 [*20 Marks*]

B4. (a) Expand in Laurent series

(i) $f(z) = \frac{e^z}{z}$, near $z_0 = 0$,

(ii) $f(z) = \frac{1}{(1-z)(2+z)}$, in power of z in the domain $|z| < 1$. (3,4)

(b) Consider $f(z) = \frac{z - \sin z}{z}$

(i) Find residue at $z = 0$,

(ii) and thus evaluate $\int_c (z)dz$, where c is a positively oriented circle $|z| = 1$. (5,2)

(c) Consider $f(z) = \frac{e^{2z}}{(z-1)^2.}$

(i) Show that $z = 1$ is a pole. Find its order.

(ii) Find residue at $z = 1$. (3,3)

QUESTION B5 [*20 Marks*]

B5. (a) Consider $f(z) = \dfrac{e^z - 1}{z}$.

(i) Show that $z = 0$ is a removable singular point.

(ii) Let $g(z) = f(z)$ for $z \neq 0$.

Define $g(0)$ to make $g(z)$ an entire function. (4,2)

(b) Apply the Residue theorem to evaluate

(i) $\displaystyle\int_c \frac{dz}{z^2+4}$, where c is a positively oriented circle $|z+i| = 2$.

(ii) $\displaystyle\int_0^\infty \frac{\cos x}{x^2+a^2}dx, \quad a > 0.$ (6,8)

END OF EXAMINATION PAPER