UNIVERSITY OF SWAZILAND

MAIN EXAMINATION, 2015/2016

BASS III, BED III, BSC III

Title of Paper : ABSTRACT ALGEBRA I

Course Number : M323

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. A formula sheet is provided on the last page.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

(a) Define each of the following.

i. A relation from a set X into a set Y. [2]

ii. A function from a set X into a set Y. [2]

iii. An equivalence relation on a set X. [3]

(b) Use the Euclidean algorithm to find gcd(512, 320) and hence find integers s and t such that

$$\gcd(512, 320) = 512s + 320t.$$

[7]

(c) i. Give the definition of a group. [4]

ii. Let $S = \mathbb{R} \setminus \{-1\}$ and define a binary operation $*$ on S by

$$a * b = a + b + ab,$$

where $a + b$ is usual addition and ab is usual multiplication. Show that $(S, *)$ is a group. [10]

(d) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 1 & 7 & 5 & 3 & 4 & 2 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 8 & 3 & 4 & 5 & 2 & 6 \end{pmatrix}$.

i. Express α and β as products of disjoint cycles. [4]

ii. Express α and β as products of transpositions and indicate whether they are even or odd permutations. [4]

ii. Compute $\beta^{-1}\alpha$. [4]

END OF SECTION A – TURN OVER

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [20 *Marks*]

(a) Let $G = \{x \in \mathbb{Q} : x > 0\}$. Define a binary operation $*$ on G by

$$a * b = \frac{ab}{2} \quad \text{for all } a, b \in G.$$

Show that $(G, *)$ is a group. [10]

(b) Let $(G, *)$ be a group and let $a, b, c \in G$. Prove each of the following.

i. $(a * b)^{-1} = b^{-1} * a^{-1}$, [6]

ii. If $a * b = a * c$, then $b = c$. [4]

QUESTION B3 [20 *Marks*]

(a) Prove: *A subset H of a group G is a subgroup of G if and only if $H \neq \emptyset$ and for $g, h \in H$, $g * h^{-1} \in H$.* [10]

(b) Let H be the subset

$$\{\rho_0 = (1), \rho_1 = (123), \rho_2 = (132)\}$$

of the symmetric group S_3.

i. Show that H is a subgroup of S_3. [6]

ii. Show that H is cyclic. [4]

QUESTION B4 [20 *Marks*]

(a) Define a relation $\sim$ on $\mathbb{Z}$ by $m \sim n$ if and only if $m \equiv n \pmod 4$.

i. Show that $\sim$ is an equivalence relation on $\mathbb{Z}$. [8]

ii. Describe the partition given by $\sim$. [4]

(b) Let $a, b, m \in \mathbb{Z}$. Show that if $\gcd(a, m) = 1$ and $\gcd(b, m) = 1$, then $\gcd(ab, m) = 1$. [8]

QUESTION B5 [20 *Marks*]

(a) Prove that every cyclic group is abelian. [7

(b) Find the number of generators of the cyclic group $\mathbb{Z}_{42}$ and then list them. [6

(c) Prove that every group of prime order is cyclic. [7

QUESTION B6 [20 *Marks*]

(a) Find all the subgroups of $\mathbb{Z}_{18}$ and give a lattice diagram. [1

(b) Let $\phi : G \to H$ be a group isomorphism and let e be the identity of G. Prove that $\phi(e)$ is the identity in H and that $[\phi(a)]^{-1} = \phi(a^{-1})$. [1

END OF EXAMINATION PAPER