UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2015/2016

BASS III, BED III, BSC III

Title of Paper : ABSTRACT ALGEBRA I

Course Number : M323

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

(a) Define each of the following.

i. A relation from a set X into a set Y.

ii. A function from a set X into a set Y.

iii. An equivalence relation on a set X.

(b) Use the Euclidean algorithm to find $\gcd(126, 45)$ and hence find integers s and t such that

$$\gcd(126, 45) = 126s + 45t.$$

(c) i. Give the definition of a group.

ii. Let

$$G = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{Q} \text{ and } ac \neq 0 \right\}.$$

Show that G with matrix multiplication is a group. [

(d) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and define a relation R on A by

$$aRb \quad \text{if } 3 \mid (a - b)$$

for $a, b \in A$.

i. List the elements of R. [

ii. Determine whether or not R is an equivalence relation on A. [

(e) Let $a, b, c \in \mathbb{Z}$. Suppose $\gcd(a, c) = 1$ and $c \mid ab$. Prove that $c \mid b$. [

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

(a) Consider the following permutations in S_6

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}.$$

Compute

i. β^{-1}
ii. β^{-2}
iii. $\alpha\beta^{-1}$

(b) Write the permutations in (a) as a product of disjoint cycles in S_6 and then as products of transpositions. Indicate whether the permutation is even or odd.

(c) Prove that every cyclic group is abelian. [

QUESTION B3 [*20 Marks*]

(a) Prove: *A subset H of a group $(G, *)$ is a subgroup of G if and only if it satisfies the following conditions.*

1. *The identity e of G is in H.*
2. *For $h_1, h_2 \in H$, $h_1 * h_2 \in H$.*
3. *For $h \in H$, $h^{-1} \in H$.*

[

(b) Let G be the group of all 2×2 matrices under addition and let

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, a + d = 0 \right\}.$$

Show that H is a subgroup of G. [

QUESTION B4 [20 *Marks*]

(a) Define a relation $\sim$ on $\mathbb{Z}$ by $m \sim n$ if and only if $m \equiv n (\text{mod } 5)$.

i. Show that $\sim$ is an equivalence relation on $\mathbb{Z}$. [8

ii. Describe the partition given by $\sim$. [4

(b) Let a and b be integers and p a prime number. Prove that if $p \mid ab$, then either $p \mid a$ or $p \mid b$. [8

QUESTION B5 [20 *Marks*]

(a) Find the number of generators of the cyclic group $\mathbb{Z}_{30}$ and then list them. [5

(b) Let $S = \mathbb{R} \setminus \{0\}$ and consider the groups $(S, +)$ and $(\mathbb{Z}, +)$ where $+$ is the usual addition. Let $G = S \times \mathbb{Z}$. Define a binary operation $*$ on G by

$$(a, m) * (b, n) = (ab, m + n).$$

i. Show that G is closed under $*$. [2]

ii. Show that $(G, *)$ is a group. [8]

(c) Prove that every group of prime order is cyclic. [5]

QUESTION B6 [20 *Marks*]

(a) i. Define a group isomorphism. [3]

ii. Let $\phi : G \to H$ be a group isomorphism and let e be the identity of G. Prove that $\phi(e)$ is the identity in H. [7]

(b) Find all the subgroups of $\mathbb{Z}_{12}$ and give a lattice diagram. [10

END OF EXAMINATION PAPER