UNIVERSITY OF SWAZILAND

EXAMINATION, 2015/2016

BASS III, B.Ed (Sec.) III, B.Sc. III

Title of Paper : Real Analysis

Course Number : M331

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

(a) Consider a set $A \subseteq \mathbb{R}$.

i. Explain what it means to say that the set A is bounded above and define $\sup(A)$ for such a set. (2

ii. Explain what it means to say that the set A is bounded below and define $\inf(A)$ for such a set. (2

(b) Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Is A bounded? Find (if they exist) $\min(A)$, $\max(A)$, $\inf(A)$, and $\sup(A)$. (6

(c) i. Give the ε, N definition for the convergence of a sequence $\{x_n\}$ to the number x. (3

ii. Explain what it means for a sequence $\{x_n\}$ to be Cauchy. (3

(d) Let $\{x_n\}$ be a sequence.

i Explain what it means to say that a series $\sum_{n=1}^{\infty} x_n$ converges to the number s. (3

ii. State the Test for Divergence for a series $\sum_{n=1}^{\infty} x_n$. (3

iii. True or false? *If* $\lim_{n\to\infty} x_n = 0$, *then the series* $\sum_{n=1}^{\infty} x_n$ *is convergent*. Explain your answer. (3

(e) Let $f : [a,b] \to \mathbb{R}$ be a function and let $c \in (a,b)$.

i. Explain what it means to say that f is continuous at c. (3

ii. Explain what it means to say that f is differentiable at c. (3

iii. True or false? *If f is continuous at c, then f is differentiable at c.* Explain your answer. (3

(f) State Riemann's integrability criterion. (3

(g) True or false? *If f is integrable on $[a,b]$, then f is continuous on $[a,b]$.* Explain your answer. (3

END OF SECTION A – TURN OVER

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [20 *Marks*]

(a) Let $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$. Show that

$$ad + bc < ac + bd.$$

(4

(b) Prove the reverse triangle inequality

$$\Big| |x| - |y| \Big| \leq |x - y| \quad \text{for } x, y \in \mathbb{R}.$$

(6

(c) Show that the square of an odd number is odd. (3

(d) Prove that there is no rational number x such that $x^2 = 2$. (7

QUESTION B3 [20 *Marks*]

(a) i. State the Cauchy convergence criterion. (3

ii. Use the Cauchy convergence criterion to show that the sequence

$$\left\{ \frac{n+1}{n} \right\}$$

is convergent. (7

(b) Consider the sequence recursively defined by

$$x_1 = \sqrt{2}, \quad x_n = \sqrt{2 + x_{n-1}}, \; n \geq 2.$$

i. Use mathematical induction to show that the sequence $\{x_n\}$ is increasing. (4

ii. Use mathematical induction to show that $x_n < 10$ for all $n \in \mathbb{N}$. (3

iii. Determine whether the sequence converges or not. Give reasons for your answer. (3

QUESTION B4 [20 *Marks*]

(a) i. State the Intermediate Value Theorem. (3

ii. Show that the equation $\sin x = 1 - x$ has a solution in the interval $[0, \pi/2]$. (3

(b) Prove: *If* $\lim_{x \to c} f(x)$ *exists, then it is unique.* (7

(c) Let $f(x) = x^2$ and let $0 < a < \infty$. Show that f is uniformly continuous on $[-a, a]$. (7

QUESTION B5 [20 *Marks*]

(a) Prove: *If* $f : I \to \mathbb{R}$ *is differentiable at* $c \in I$, *then* f *is continuous at* c. (7

(b) State the Fundamental Theorem of Calculus. (3

(c) Let $f : [0, 2] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & \text{if } x < 1, \\ \frac{1}{2}, & \text{if } x = 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Show that f is Riemann integrable and find $\int_0^2 f(x)\, dx$. (1

QUESTION B6 [20 *Marks*]

(a) Prove: *If* $\sum_{n=1}^{\infty} x_n$ *is convergent, then* $\lim_{n \to \infty} x_n = 0$. (7

(b) Explain what it means to say that a series $\sum_{n=1}^{\infty} x_n$ *converges absolutely.* (3

(c) True or False? *If the series* $\sum_{n=1}^{\infty} x_n$ *is convergent, then it converges absolutely.* Explain your answer. (3

(d) Prove: *If the series* $\sum_{n=1}^{\infty} x_n$ *converges absolutely, then it is convergent.* (7

END OF EXAMINATION PAPER