UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2015/2016

BASS III, B.Ed (Sec.) III, B.Sc. III

Title of Paper : Real Analysis

Course Number : M331

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

(a) Let $A = \{x \in \mathbb{R} : x^2 - x - 1 < 0\}$. Find $\sup(A)$ and $\inf(A)$. Does $\min(A)$ exist? Does $\max(A)$ exist?

(b) i. Define a Cauchy sequence.

ii. State the Cauchy convergence criterion for a sequence.

ii. Define a Cauchy series.

(c) i. Given a bounded function f and a partition P of the interval $[a,b]$, give the definitions of the *Riemmann upper sum* $U(P,f)$ and the *Riemann lower sum* $L(P,f)$ of f, being careful to explain each symbol used.

ii. State the Riemann integrability criterion.

iii. Prove or disprove: *If f is integrable on $[a,b]$, then f is continuous on $[a,b]$.*

(d) i. Explain what it means to say that a series $\sum x_n$ converges absolutely. (

ii. Prove or disprove: *If a series is convergent, then it converges absolutely.* (

(e) i Give the ε, δ definition for a function f to have the limit L as x approaches a number c. (

ii. Explain what it means to say a function f is differentiable at c. (

(f) State the Intermediate Value Theorem. (

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [20 *Marks*]

(a) Show that for $x, y \in \mathbb{R}$, the following inequality

$$xy \leq \frac{x^2+y^2}{2}$$

holds, with equality if $x = y$. Hence deduce the arithmetic-geometric mean inequality

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

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(b) Prove the reverse triangle inequality

$$\Big||x|-|y|\Big| \leq |x-y| \quad \text{for } x, y \in \mathbb{R}.$$

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(c) Use the order axioms to show that $n^2 > n$ for all $n \in \mathbb{N}$ with $n > 1$. (

(d) Prove that there is no rational number x such that $x^2 = 2$. (

QUESTION B3 [20 *Marks*]

(a) Use the ε, N definition to show that

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

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(b) Let $\{x_n\}$ be the sequence defined recursively by

$$x_1 = 2, \quad x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}.$$

i. Use mathematical induction to show that $x_n \geq 0$ for all $n \in \mathbb{N}$. (
ii. Use mathematical induction to show that the sequence $\{x_n\}$ is non-increasing. (
iii. Determine whether the sequence converges or not. Give reasons for your answer. (

QUESTION B4 [20 *Marks*]

(a) Show that the equation
$$\ln x = 2 - x$$
has a solution in the interval $[1, e]$. (

(b) Let $f : [a, b] \to \mathbb{R}$ be a function. Explain what each of the following statements means.

i. f is right-differentiable at $c \in [a, b)$. (

ii. f is left-differentiable at $c \in (a, b]$. (

(c) Give an example of a function that is left-differentiable and right-differential at a point c in its domain but not differentiable at c. (

(d) Let $f(x) = 10x - 9$. Use an ε, δ argument to show that $\lim_{x \to 2} f(x) = 11$. (

QUESTION B5 [20 *Marks*]

(a) Let $f(x) = x^2$. Show that f is uniformly continuous on $[0, 1]$. (

(b) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x^3$ and let $P = \{0, 0.1, 0.4, 1\}$. Find $U(P, f)$ and $L(P, f)$. (

(c) Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$

Show that f is Riemann integrable and find $\int_{-1}^{1} f(x)dx$. (

QUESTION B6 [20 *Marks*]

(a) Prove or disprove: *If* $\sum x_n$ *and* $\sum y_n$ *are convergent series, then* $\sum x_n y_n$ *is convergent.* (

(b) Prove: *Let* $\{x_n\}$ *and* $\{y_n\}$ *convergent sequences. Then*
$$\lim_{n \to \infty} (x_n + y_n) = \lim_{n \to \infty} x_n + \lim_{n \to \infty} y_n.$$
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(c) Prove: *If the series* $\sum_{n=1}^{\infty} x_n$ *converges absolutely, then it is convergent.* (

END OF EXAMINATION PAPER