University of Swaziland

Final Examination 2015/2016

B.A.S.S. , B.Sc, B.Ed

Title of Paper : Numerical Analysis II

Course Number : M411

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Show all your working.
3. Start each question on a fresh page.
4. Non programmable calculators may be used (unless otherwise stated).
5. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A

A1. (a) State the Weierstrass theorem for the approximation of functions with polynomials on intervals. [3]

(b) Let $w(x), \phi_0(x), \phi_1(x), \ldots, \phi_n(x)$ be functions that are defined on an interval $I \subseteq \mathbb{R}$. Precisely explain the following statements.

i. The set $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ is orthogonal on I with respect to weight function $w(x)$. [3]

ii. $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ is linearly independent on I. [3]

(c) Use the Gram-Schmidt procedure to determine the set $\{L_0(x), L_1(x)\}$ of (Laguerre) polynomials that are orthogonal on $(0, \infty)$ with respect to the weight function $w(x) \equiv e^{-x}$ and $L_0(x) \equiv 1$. [5]

A2. (a) Solve the initial value problem (IVP)

$$y'(x) = 3e^{-4x} - 2y, \quad y(0) = 1$$

for $y(0.1)$ using one step of each of the following methods.

i. Euler method [3]

ii. Taylor series method of order 2. [3]

iii. Runge-Kutta method of order 2. [3]

(b) A certain multi-step method has

$$\rho(r) = 2 - 3r + r^2 \text{ and } \sigma(r) = -\frac{3}{2} + \frac{1}{2}r$$

as its first and second characteristic polynomial respectively. State the Dahlquist Equivalence Theorem for convergence and use it to analyze the convergence of the difference equation. [5]

A3. (a) Let Γ be the boundary of an *open* and *connected* region $\Omega \subseteq \mathbb{R}^2$. A boundary value problem consists of finding $u(x, y)$ satisfying

$$\begin{aligned} \nabla^2 u &= 0 \text{ on } \Omega, \\ u &= f \text{ on } \Gamma \end{aligned}$$

where $f(x, y)$ is a given function. Derive the *five point formula* for approximating u at any grid point in Ω. Be careful to explain any notation used. [5]

(b) Let a be a constant, let $j = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$. The upstream scheme

$$U_j^{n+1} = U_j^n - \alpha(U_j^n - U_{j-1}^n), \tag{1}$$

where $\alpha = a\frac{\Delta t}{\Delta x}$, can be used for approximating the advection equation

$$u_t + au_x = 0 \tag{2}$$

i. Derive equation (1). [5]

ii. Specify the range of values of α for which the upstream scheme converges. Justify your answer. [2]

SECTION B

B4. (a) Find a linear function $p_1(x)$ that approximates $\ln(x-2)$ on the closed interval $[3, 4]$ in the least squares sense. [10]

(b) Find a linear function $p_1(x)$ that best fits the data

j	0	1	2	3
x_j	3	5	8	10
y_j	8.3	11.3	14.4	15.9

in the least squares sense. [10]

B5. (a) Show that the Chebyshev polynomials $\{T_0(x), T_1(x), \ldots\}$ are orthogonal on $[-1, 1]$ with respect to the weight function $\frac{1}{\sqrt{1-x^2}}$. [10]

(b) State the Gram-Schmidt process for orthogonal polynomials on a given interval $[a, b]$ with respect to a weight function $w(x)$. [10]

B6. Use a single step of the Runge-Kutta method of order 4 to solve the Initial Value problem

$$y'' - 3y' + 2y = 6e^{-x},\ 0 \le x \le 1,\ y(0) = y'(0) = 2,$$

for both $y(0.1)$ and $y'(0.1)$. [20]

B7. Consider the differential problem;

$$\begin{aligned} u_t(x,t) =& u_{xx}(x,t),\ 0 < x < 1,\ t > 0, \\ u(0,t) =& 0,\ u_x(1,t) = u(1,t) - 1, t > 0, \\ u(x,0) =& \sin(\pi x),\ 0 \leq x \leq 1. \end{aligned}$$

Suppose that an approximate solution to this problem is determined by replacing u_t with a **forward** difference, and that both u_x and u_{xx} are replaced by **central** differences.

(a) Show that the resulting finite difference equations may be written in matrix form as

$$\mathbf{u}^{n+1} = B\mathbf{u}^n + \mathbf{v},\ \text{ where } n = 1, 2, \ldots$$

Identify the square matrix B, and the vectors $\mathbf{u}_j$ and $\mathbf{v}$. [12]

(b) Compute the leading terms of the truncation error for this numerical scheme. [8]

B8. (a) Consider the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} =& 0,\ 0 \leq x \leq 2,\ 0 \leq y \leq 3, \\ u(x,0) =& x/2,\ u(x,3) = 1,\ 0 \leq x \leq 2, \\ u(0,y) =& y/3,\ u(2,y) = 1,\ 0 \leq y \leq 3. \end{aligned}$$

Use a finite difference method known as "*the* 5 *point formula*" with a uniform grid on S to approximate both $u(1,1)$ and $u(1,2)$. [10]

(b) Determine a sufficient condition for convergence of the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0,$$

for approximating the advection equation

$$u_t + au_x = 0,\ \ u(x,0) = f(x),\ \text{ where } a > 0 \text{ is given.}$$ [10]

END OF EXAMINATION