University of Swaziland

Final Examination, December 2015

B.A.S.S. , B.Sc, B.Eng, B.Ed

Title of Paper : Partial Differential Equations

Course Number : M415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Non programmable calculators may be used (unless otherwise stated).
5. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

1.1. Write short notes explaining the following concepts

(a) order of a PDE, [2]

(b) linearity, [4]

(c) well posedness of a PDE, [4]

(d) boundary conditions. [6]

1.2. Determine the region in which the given partial differential equation

$$u_{xx} - \sqrt{\frac{(x-2)^2 + y^2}{8}}\, u_{xy} + 2u_{yy} = e^{3x}$$

is hyperbolic, parabolic or elliptic. $(u = u(x, y))$. [6]

1.3. Solve the following partial differential equations, $(u = u(x, y))$.

(a) $(2xy - 1)u_x + (u - 2x^2)u_y = 2(x - uy)$, [6]

(b) $(1 - x)u_x + u_y = 0$. [7]

1.4. Assuming $1 + xy \neq 0$, show that $u(x, y) = F\left(\frac{x}{1 + xy}\right)$ is a solution of the following PDE

$$x^2 u_x + u_y = 0.$$

[5]

SECTION B: ANSWER ANY 3 QUESTIONS

2. Solve the following partial differential equations using the method of Laplace transforms

   (a) $u_{xt} + \sin t = 0, \quad u(x,0) = x,\ u(0,t) = 0.$ [10]

   (b) $xu_x + u_t = xt, \quad u(x,0) = 0,\ u(0,t) = 0.$ [10]

3. Consider the following partial differential equation

$$2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0.$$

   (a) Classify the partial differential equation as hyperbolic, parabolic or elliptic. [3]

   (b) Reduce the equation into its canonical form and hence find the general solution. [17]

4. Consider the function

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0; \\ 0, & x = 0; \\ +1, & 0 < x \leq \pi. \end{cases} \qquad f(x + 2\pi) = f(x).$$

   (a) Find the fourier series expansion. [12]

   (b) Use Parseval's identity to find the value of the sum

$$\sum_{n-1}^{\infty} \frac{1}{(2n-1)^2}$$

[8]

5. Find the particular solutions for the following partial differential equations

   (a) $yu_y - x^2u_y = xy, \quad u = x^2$ on $3y^2 = 2x^3$. [10]

   (b) $u_{xy} = 1, \quad u = 0$ and $u_x = 0$ on $x + y = 0$. [10]

6. Show that the initial value problem with non-homogeneous boundary conditions

$$\begin{aligned} &u_t - u_{xx} = 0, \quad 0 < x < L, \quad t > 0 \\ &u(0,t) = T_1, \quad t \geq 0 \\ &u(L,t) = T_2, \quad t \geq 0 \\ &u(x,0) = f(x), \quad 0 \leq x \leq 1 \end{aligned}$$

can be transformed to an initial value problem with homogeneous boundary conditions. Hence find the general solution of initial value problem using separation of variables. [20]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{d^n f}{dt^n}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$