UNIVERSITY OF SWAZILAND

FINAL EXAMINATION, 2015/2016

B.Sc. IV, BASS IV, BED. IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked**.
2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

A1. (a) Give a definition of the following

(i) an integral domain

(ii) a divisor ring

(iii) a field (2,2,2)

(b) Prove that a finite integral domain is a field. (10)

(c) Give an example of an integral domain that is not a field (4)

A2. (a) Define

(i) an ideal H in a ring R

(ii) a divisor of zero in a ring R

(iii) the characteristic of a ring R

(iv) unity in a ring R

(v) a unit in a ring R (2,2,2,2

(b) Prove that in a ring $\mathbb{Z}_n$

(i) the divisor of zero are those elements that are not relatively prime to n.

(ii) the elements that are relatively prime to n cannot be divisors of zero. (5,5)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

(a) Find the greatest common divisor (gcd) of the polynomials $f(x) = x^4 + 4x^3 + 7x^2 + 6x+2$ and $g(x) = x^3+4x^2+7x+4$ over Q and express the gcd as a linear combination of $f(x)$ and $g(x)$. (7)

(b) Prove that if R is a ring with unity and N an ideal of R containing a unit, then $N = R$. (7)

(c) Describe all units in each of the following rings

(i) $\mathbb{Z}_7$

(ii) $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$ (6)

QUESTION B4 [*20 Marks*]

(a) State Eisentein's criterion for irreducibility. (3)

(b) Use Eisentein's criterion to show that $f(x) = 26x^5 - 5x^4 + 25x^2 - 10$ is irreducible over Q. (4)

(c) Find all zeros of $x^3 + 2x + 2$ in $\mathbb{Z}_7$. (5)

(d) Find the quotient $q(x)$ and the remainder $r(x)$ when the polynomial $f(x) = 4x^5 + 3x^4 + x^3 + 2x^2 + 1$ is divided by $g(x) = 3x^2 + x + 3$ in $\mathbb{Z}_5$. (8)

QUESTION B5 [*20 Marks*]

(a) Which of the following sets is a ring with the usual operations of addition and multiplication? In each case either prove that it is a ring or explain why it is not.

(i) The set $\{1, -1, i, -i\}$

(ii) The set $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a.b \in \mathbb{Z}\}$ (6,6)

(b) Show, that $\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic as rings. (8)

QUESTION B6 [*20 Marks*]

(a) Show that the polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$ (6)

(b) Let α be a zero of $x^2 + x + 1$ in the extension field of $\mathbb{Z}_2, E = \mathbb{Z}(\alpha)$

(i) Write down all the elements of $\mathbb{Z}_2(\alpha)$

(ii) Construct the multiplication for $\mathbb{Z}_2(\alpha)$, showing the inverse for each non-zero element. (8)

QUESTION B7 [*20 Marks*]

(a) Show that if D is an integral domain then the ring $D[x]$ of polynomials is also an integral domain. (7)

(b) The polynomial $x^4+2x^3+x^2+x+1$ has a linear factor in $\mathbb{Z}_3[x]$. Find the factorization into irreducible polynomial in $\mathbb{Z}_3[x]$ (7)

(c) Show that $\alpha = \sqrt{1-\sqrt{2}}$ is algebraic over $\mathbb{Q}$. Find the minimum polynomial and the degree of α

(i) Over $\mathbb{R}$

(ii) Over $\mathbb{Q}$. (3,3)

END OF EXAMINATION PAPER