UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2015/2016

B.Sc. IV, BASS IV, BED. IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked.**
2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

A1. (a) Prove that every field is a integral domain (6)

(b) Which of the following are rings with the usual addition and multiplication

(i) $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$

(ii) $M_2(\mathbb{R})$ with zero determinant (8)

(c) Mark each of the following true or false

(i) Every finite integral domain is a field

(ii) Every division ring is commutative

(iii) $\mathbb{Z}_6$ is not an integral domain (6)

A2. (a) Use Fermat's theorem to computer the remainder when 8^{123} is divided by 13 (6)

(b) For each of the following, find irred (α, Q) and $\deg(\alpha, Q)$.

(i) $\sqrt{3} + i$

(ii) $\sqrt{\frac{1}{5} + \sqrt{7}}$ (6)

(c) Show that if a polynomial $f(x) \in \mathbb{Z}[x]$ is reducible over $\mathbb{Q}$ then its also reducible over $\mathbb{Z}[x]$. (8)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

(a) Show that for a field F, the set of all matrices of the form $\begin{pmatrix} a_{11}, & a_{12} \\ 0 & 0 \end{pmatrix}$ $a_{ij} \in F$ is a right ideal but not a left ideal of $M_2(F)$ (6)

(b) Let $\varphi_\alpha : \mathbb{Z}_7[x] \to \mathbb{Z}_7$. Evaluate each of the following for the indicated evaluation homomorphism

(i) $\varphi_2(3x^{79} + 5x^{53} + 2x^{43})$

(ii) $\varphi_3[(x^3 + 2)(4x^2 + 3)(x^7 + 3x^2 + 1)$ (10

(c) Show that if D is an integral domain, then $D(x)$ is also an integral domain. (4)

QUESTION B4 [*20 Marks*]

(a) Suppose F is a filed, f is an irreducible polynomial over F and g, h are polynomials over F such that f divides gh show that either f divides g or f divides h. (10

(b) Define an ideal N of a ring R (2)

(c) Find all ideals of $\mathbb{Z}_{10}$ and all maximal ideals of $\mathbb{Z}_{18}$. (5)

QUESTION B5 [*20 Marks*]

(a) Find the greatest common divisor of the polynomials $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2$ and $g(x) = x^3 + 4x^2 + 7x + 4$. over $\mathbb{Q}$ and express it as a linear combination of $f(x)$ and $g(x)$. (8)

(b) Prove that if R is a ring with unity and N is an ideal of R containing a unit, then $N = R$. (6)

(c) Describe all units in each of the following rings

(i) $\mathbb{Z}_7$

(ii) $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}_3$ (6)

QUESTION B6 [*20 Marks*]

(a) State whether or not each of the given function v, is an Euclidea evaluation for the given integral domain

(i) $v(n) = n^2$ for a non-zero $n \in \mathbb{Z}$,

(ii) $v(a) = s_0$ for a non-zero values $a \in \mathbb{Q}$ (8)

(b) State Knonecker's theorem [Do Not Prove]

(c) Given that every element β of $\beta = F(\alpha)$ can be uniquely expressed in the form $\beta = b_0 + b_1\alpha^2 + \cdots + b_{n-1}\alpha^{n-1}$ where each $b_i \in F, \alpha$ algebraic over the field F and $(\alpha, F) \geq 1$. Show that if F is finite with q elements, then $F(\alpha)$ has q^n elements. (8)

QUESTION B7 [*20 Marks*]

(a) Classify each of the given $\alpha \in$ as algebraic or transcendal over the given field F. If α is algebraic over F find $\deg(\alpha, F)$

(i) $\alpha = 1 + i, F = \mathbb{Q}$

(ii) $\alpha = \sqrt{\pi}, F = \mathbb{Q}[\pi]$

(iii) $\alpha = \pi^2, F = \mathbb{Q}$

(iv) $\alpha = \pi^2, F = \mathbb{Q}(\pi^3)$

(v) $\alpha = \pi^2, F = \mathbb{Q}(\pi)$ (10

(b) Show that the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ is NOT a field (5)

(c) Find a polynomial of degree > 0 in $\mathbb{Z}_4[x]$ that is a unit. (5)

END OF EXAMINATION PAPER