UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2015/2016

B.Sc. / B.Ed. / B.A.S.S. IV

TITLE OF PAPER	:	METRIC SPACES
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ALL QUESTIONS IN SECTION A. 3. ANSWER ANY THREE QUESTIONS IN SECTION B.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

SECTION A

QUESTION 1

Let (X, d) be a metric space. Define the following:

(a) the distance from $x \in X$ to a subset $A \subset X$; [2]

(b) the diameter of $A \subset X$; [2]

(c) the distance between two subsets, A and B, of X; [2]

(d) a bounded subset $A \subset X$; [2]

(e) a bounded mapping g from a nonempty set Y to X; [2]

(f) a convergent sequence (x_n) in X; [2]

(g) a Cauchy sequence (x_n) in X; [2]

(h) a subspace (Y, d_Y) of (X, d); [2]

(i) an open ball $B(a, r)$ in (X, d); [2]

(j) an open subset F of X. [2]

QUESTION 2

(a) Let d be the function on $\mathbb{R}^2$ defined by

$$d(x, y) = c_1|x_1 - y_1| + c_2|x_2 - y_2|,$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and $c_1, c_2 \in \mathbb{R}$. Prove that $(\mathbb{R}^2, d)$ is a metric space. [12]

(b) A translation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a map given by $T(x) = (x_1 + a, x_2 + b)$ for some fixed point $(a, b) \in \mathbb{R}^2$, where $x = (x_1, x_2) \in \mathbb{R}^2$. Prove that the Euclidian metric d on $\mathbb{R}^2$ is translation invariant, in the sense that for any two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, we have

$$d_2(T(x), T(y)) = d_2(x, y).$$

[3]

(c) If a sequence (x_n) is convergent and has limit x, prove that every subsequence $(x_{n_k})_{k \geq 1}$ of (x_n) is convergent and has the same limit x. [5]

SECTION B

QUESTION 3

(a) Let $X = \mathcal{C}[-2, 2]$, and let $x(t) = t$ and $y(t) = t^2$ for $t \in [-2, 2]$. Find $d(x, y)$ in $\mathcal{C}[-2, 2]$, where d is the

(i) uniform metric, [9]

(ii) L_1–metric, [3]

(iii) L_2–metric. [3]

(b) Suppose that $f, g : X \longrightarrow \mathbb{R}$ are both continuous. Show that the function $h : X \longrightarrow \mathbb{R}$ defined by

$$h(x) = 6f(x) - 5g(x)$$

is continuous. [5]

QUESTION 4

Let $A \subset \mathbb{R}^2$ be the region bounded by the unit disc centered at the origin, and let $x = (4, 4)$. Find $d(x, A)$ in $\mathbb{R}^2$ with each of the following metrics:

(a) the London (or UK)-rail metric; [3]

(b) the Max metric; [4]

(c) the Chicago metric; [8]

(d) the New York metric; [3]

(e) the Raspberry pickers' metric. [2]

QUESTION 5

(a) Let (X, d) be a metric space, and let (x_n) and (y_n) be two sequences in X such that (y_n) is a Cauchy sequence and $d(x_n, y_n) \to 0$ as $n \to \infty$.Prove that:

(i) (x_n) is a Cauchy sequence in X; [5]

(ii) (x_n) converges to a limit x in X if and only if (y_n) also converges to x in X. [5]

(b) Prove that every Cauchy sequence in a metric space (X, d) is bounded. [4]

(c) Let (X, d) be a metric space, and let d' be the metric on X defined by

$$d'(x, y) = \min\{1, d(x, y)\}.$$

Prove that (x_n) i a Cauchy sequence in (X, d) if and only if (x_n) is a Cauchy sequence in (X, d'). [6]

QUESTION 6

(a) Given a function $f : (X, d_1) \longrightarrow (X, d_2)$,

(i) When is f said to be continuous in the $\varepsilon - \delta$ sense?

(ii) Give an equivalent definition in terms of open sets.

(iii) Assuming f is continuous at x_0, prove that

$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0).$$

[14]

(b) Prove that the function $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\pi(x, y) = x$ is continuous when $\mathbb{R}^2$ and $\mathbb{R}$ are equipped with their usual metrics. Is π uniformly continuous? Justify your answer. [6]

QUESTION 7

(a) Let (X, d) be a metric space and (x_n) be a sequence in X. What is meant by saying that (x_n) is *convergent*? [2]

(b) Decide whether or not the following sequences are convergent in the usual (Euclidean) metric on $\mathbb{R}^2$:

(i) $x_n = \left(\frac{n^2}{3n^2+1}, \frac{1}{n+2}\sin(\frac{n\pi}{2})\right)$,

(ii) $x_n = (3^{-2n}, (-1)^n \exp(\frac{1}{n}))$. [4,4]

(c) Consider $\mathbb{R}^2$ with the New York metric. Let $(x^{(n)})_{n\geq 1}$, where $x^{(n)} = (x_1^{(n)}, x_2^{(n)})$, be a sequence of points in $\mathbb{R}^2$. Prove that $(x^{(n)})_{n\geq 1}$ converges to $x = (x_1, x_2) \in \mathbb{R}^2$ if and only if either

1. $x_1^{(n)} = x_1 \, \forall \, n \in \mathbb{N}$ and $x_2^{(n)} \to x_2$, or
2. $x_1^{(n)} \neq x_1$ for some $n \in \mathbb{N}$, and $x_1^{(n)} \to x_1, x_2^{(n)} \to 0$, and $x_2 = 0$. [10]

END OF EXAMINATION