UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2015/2016

B.Sc. / B.Ed. / B.A.S.S. IV

TITLE OF PAPER	:	METRIC SPACES
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ALL QUESTIONS IN SECTION A. 3. ANSWER ANY THREE QUESTIONS IN SECTION B.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

SECTION A

QUESTION 1

(a) What is meant by "(X, d) is a *metric space*?" [3]

(b) A translation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a mapping given by $T(x) = (x_1 + a, x_2 + b)$ for some fixed point $(a, b) \in \mathbb{R}^2$, where $x = (x_1, x_2) \in \mathbb{R}^2$. Prove that the Euclidian metric d_2 on $\mathbb{R}^2$ is translation invariant, in the sense that for any two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, we have

$$d_2(T(x), T(y)) = d_2(x, y).$$

[3]

(c) Suppose that (x_n) converges to x in $C\,[a, b]$ in the uniform metric. Explain what is meant by *pointwise convergence* of a sequence (x_n) in $C\,[a, b]$. Show that $\{x_n\}$ converges to x pointwise. [4]

(d) Define compactness of a metric space in terms of

(i) open coverings, [2]

(ii) sequences. [1]

(e) Explain what is meant by a contraction of a metric space, and state without proof the Contraction Mapping Theorem. [3]

(f) (i) What is a Lebesgue number for a given open cover of a metric space? [1]

(ii) State without proof Lebesgue's Covering Lemma. [3]

QUESTION 2

Show that infinitely many metrics can be defined on a set with more than one element. (**Hint:** Given a metric space (X, d_1), show that $d_2(x, y) = \dfrac{d_1(x, y)}{1 + d_1(x, y)}$, $\forall\, x, y \in X$, is a metric on X, and then apply induction on $n \in \mathbb{N}$). [20]

SECTION B

QUESTION 3

(a) Can you find a metric space (X, d) where:

  (i) The interval $[0, 1]$ is both open and closed? [2]

  (ii) The interval $[0, \frac{1}{2}]$ is open but not closed? [2]

  Justify your answer in each case.

(b) Describe open balls $B(a, 3)$, where $a = (2, 3)$ in $\mathbb{R}^2$ with respect to the following metrics:

  (i) the Chicago metric; [4]

  (ii) the London (or UK)-rail metric; [4]

  (iii) the New York metric; [4]

  (iv) the Raspberry pickers' metric. [4]

QUESTION 4

Let $A \subset \mathbb{R}^2$ be the region bounded by the unit disc centered at the origin. Find $\text{diam}(A)$ with each of the following metrics:

(a) the Max metric; [4]

(b) the Chicago metric; [5]

(c) the London (or UK)-rail metric; [3]

(c) the New York metric; [4]

(d) the Raspberry pickers' metric. [4]

QUESTION 5

(a) Give two alternate definitions of connectedness of a subset M of a metric space X. [4]

(b) Let $X = \mathcal{C}[0,1]$, the set of all continuous functions on $[0,1]$, and let d be the metric on X defined by

$$d(f,g) = \int_0^1 |f(x) - g(x)| \, dx.$$

For each $n \in \mathbb{N}$, define f_n by $f_n(x) = x^n$ for all $x \in [0,1]$.

(i) Show that the sequence (f_n) converges in X, and find its limit f. [3]

(ii) Show that the function f in Part (i) is not the pointwise limit of the sequence (f_n). [3]

(c) Let d be the metric on $X = \mathcal{C}[a,b]$ defined by

$$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Let (f_n) be a sequence in $\mathcal{C}[a,b]$, and suppose that (f_n) converges uniformly on $[a,b]$ to some function f.

(i) Prove that f is continuous on $[a,b]$, and hence show that (f_n) converges in (X,d). [5]

(ii) Prove that $\displaystyle\int_a^b f_n(x) \, dx \longrightarrow \int_a^b f(x) \, dx$ as $n \to \infty$. [5]

QUESTION 6

(a) Prove that the sequence

$$(x^{(n)})_{n\geq 1} = \left(2\frac{1}{2}, 1\right), \left(2\frac{3}{4}, \frac{1}{2}\right), \left(2\frac{7}{8}, \frac{1}{3}\right), \ldots, \left(3 - \frac{1}{2^n}, \frac{1}{n}\right), \ldots$$

is not convergent in $\mathbb{R}^2$ equipped with the London (or UK)-rail metric. [10]

(b) Let f be the function $f : C[0,1] \longrightarrow \mathbb{R}$ defined for $x \in C[0,1]$ by $f(x) = x(0)$. Show that f is not continuous with respect to the L_1 metric on $C[0,1]$ (and the usual metric on $\mathbb{R}$) by considering the functions $x_n(t)$ given by

$$x_n(t) = \begin{cases} (n-1)t & \text{if } 0 \leq t \leq \frac{1}{n} \\ 1 - t & \text{if } \frac{1}{n} \leq t \leq 1 \end{cases}$$

(**Hint** Sketch the functions $x_n(t)$ and consider their limit in the L_1 metric). [7]

(c) Let (X, d) be a metric space with the metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 3 & \text{if } x \neq y. \end{cases}$$

Show that any Cauchy sequence in X is eventually constant, and deduce that (X, d) is complete. [3]

QUESTION 7

(a) Let X be a metric space. When is a subset $M \subseteq X$ said to be:

(i) bounded;

(ii) totally bounded. [3]

(b) Prove that in $\mathbb{R}$ with the usual metric, the notions of boundedness and total boundedness are equivalent. [5]

(c) Show that a compact set is closed and bounded. [6]

(d) Which of the following sets is compact? Give reasons.

(i) $\{(x, y) : 0 \leq x \leq y \leq 1\}$ in $\mathbb{R}^2$,

(ii) $\{1, \frac{1}{3}, \frac{1}{3^2}, \ldots, \frac{1}{3^n}, \ldots\}$ in $\mathbb{R}$, where $n \in \mathbb{N}$. [6]

END OF EXAMINATION