UNIVERSITY OF SWAZILAND

FINAL EXAMINATION, MAY 2017

BSc III, BEd III, BEng III, BASS III

Title of Paper : Vector Analysis

Course Number : M312

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of NINE (9) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B5) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A

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Answer ALL Questions in this section

QUESTION A1

a. Evaluate

i. $\displaystyle\int_{-\infty}^{\infty} x^2 e^{-2x^2}\,dx$ [4]

ii. $\displaystyle\int_0^1 \Big(-\ln x\Big)^5 dx$ [2]

b. Using the generating function

$$G(x,t) = \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} t^n T_n(x),$$

find the Tchebyshev polynomial $T_2(x)$. [4]

QUESTION A2

Given that $\mathbf{F} = (x + xz^2)\mathbf{i} + x\cos y\mathbf{j} + 3xe^{-yz}\mathbf{k}$ find

a. $\nabla \cdot \mathbf{F}$ [3]

b. $\nabla \times \mathbf{F}$ [5]

c. $\dfrac{\partial^2 \mathbf{F}}{\partial x \partial z}$ at the point $(1, 0, 1)$ [6]

QUESTION A3

Find the equation of the plane determined by $P(1,1,1)$, $Q(1,-3,0)$ and $R(2,2,2)$. [8]

QUESTION A4

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If $\mathbf{A} = (3x - y)\mathbf{i} + xz\mathbf{j} + (yz - 1)\mathbf{k}$, evaluate $\int_C \mathbf{A} \cdot d\mathbf{r}$ along the $x = t, y = t^2$, $z = t^3$ from $(0, 0, 0)$ to $(1, 1, 1)$ [8]

Section B

Answer and 3 Questions in this section

QUESTION B1

a. The generating function of Hermite polynomials is given by

$$G(x,h) = e^{-h^2+2xh} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H_n(x).$$

By differentiating this with respect to h, derive the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$ [10]

b. Consider the Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

i. By making the substitution $y = \dfrac{x}{1-x}$, show that

$$B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy.$$ [8]

ii. Hence, or otherwise, evaluate

$$\int_0^\infty \frac{\sqrt{y}}{(1+y)^2}\,dy.$$ [2]

QUESTION B2

a. Find the directional derivative of $\phi(x,y,z) = e^{-(x+y)^2} + z^2(x+y)$ at the point $(2,-2,1)$ in the direction $-2\mathbf{i}+\mathbf{j}-\mathbf{k}$ [5]

b. Consider the vector field

$$\mathbf{F} = (yz+y+z+2x)\mathbf{i} + (xz+4y+x)\mathbf{j} + (xy+x-1)\mathbf{k}$$

i. Show that $\mathbf{F}$ is a conservative force field [4]

ii. Find the scalar potential Φ for the vector field $\mathbf{F}$ [8]

iii. Find the work done in moving an object in the field from $(1,-2,1)$ to $(3,1,4)$ [3]

QUESTION B3

a. The acceleration of a particle at any time $t \geq 0$ is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 6\cos 2t\mathbf{i} - 4\sin 2t\mathbf{j} + 8t\mathbf{k}.$$

Find the velocity $\mathbf{v}$ given that $\mathbf{v}(0) = 0$. [3]

b. Find the outward flux $\iint_S \mathbf{F}\cdot\mathbf{n}\, ds$ where $\mathbf{F} = 2xy\mathbf{i} + e^z\mathbf{j} + x^3\mathbf{k}$ and S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. [7]

c. Use Green's Theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

to evaluate $\oint_C y^3\,dx - x^3\,dy$ where C is a positively oriented circle of radius 2 centered at the origin. [10]

QUESTION B4

a. Given the vectors $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ prove that

i. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$ [10]

ii. $\nabla \times (\phi\mathbf{A}) = (\nabla\phi) \times \mathbf{A} + \phi(\nabla \times \mathbf{A})$ [10]

QUESTION B5

a. Evaluate the line integral

$$\oint_C y\,dy - x\,dy$$

directly, where C is the circle with centre at the origin and radius 1. [8]

(Hint: Parametrize C)

b. Consider the spherical coordinate system defined by

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\varphi,\ z = r\cos\theta.$$

i. Derive the scale factors h_r, h_θ and h_ϕ. [9]

ii. Find the unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

END OF EXAMINATION PAPER