UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, JULY 2017

BSc III, BEd III, BEng III, BASS III

Title of Paper : Vector Analysis

Course Number : M312

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of NINE (9) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B5) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A

Answer ALL Questions in this section

QUESTION A1

a. Evaluate

i. $\displaystyle\int_0^{\frac{1}{2}\pi} \sin^6\theta\cos^2\theta\,\mathrm{d}\theta$ [4]

ii. $\displaystyle\int_0^{\infty} \frac{x\,\mathrm{d}x}{1+x^4}$ [3]

b. Show that the set of functions $\{1, 2x, 4x^2-2\}$ is orthogonal in the interval $(-\infty,\infty)$ with respect to the weight function $w(x)=e^{-x^2}$. [3]

QUESTION A2

Given that

$$\mathbf{A} = xyz^2\mathbf{i} + 2x^3y\mathbf{j} - yz\mathbf{k}$$

find

a. $\dfrac{\partial^2\mathbf{A}}{\partial x\partial y}$ [4]

b. $\nabla\cdot\mathbf{A}$ [3]

c. $\nabla(\nabla\cdot\mathbf{A})$ at the point $(2,-1,0)$ [5]

QUESTION A3

a. Find the directional derivative of $\phi = x^2y^2z - 3xz^3$ at $(1,1,-1)$ in the direction $2\mathbf{i}-3\mathbf{j}+6\mathbf{k}$. [5]

b. Find an equation for the plane to the surface $x^2y^2z - 3xz^3 = 2$ at the point $(1,1,-1)$. [3]

QUESTION A4

The acceleration of a particle at any time $t \geq 0$ is given by

$$\mathbf{a} = e^{-t}\mathbf{i} + 2t\mathbf{j} + 2\sin t\mathbf{k}.$$

If the velocity $\mathbf{v}$ and the displacement $\mathbf{r}$ are zero at $t = 0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time. [10]

Section B

Answer and 3 Questions in this section

QUESTION B1

a. Show that, for a Gamma distribution with parameters α and β, the k-th moment is given by

$$\mathbb{E}(X^k) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^k \cdot x^{\alpha-1}e^{-\beta x}\mathrm{d}x = \frac{\Gamma(\alpha+k)}{\beta^k\Gamma(\alpha)}. \qquad [8]$$

Hence, deduce that the mean and variance are given by

i. $\mathbb{E}(X) = \dfrac{\alpha}{\beta}$ [2]

ii. $\mathrm{Var}(X) = \mathbb{E}(X^2) - \left[\mathbb{E}(X)\right]^2 = \dfrac{\alpha}{\beta^2}$, [3]

respectively.

b. The generating function of Legendre polynomials is given by

$$G(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} = \sum_{n=0}^{\infty} h^n P_n(x).$$

By evaluating

$$2h\frac{\partial G}{\partial h} + G$$

show that

$$\frac{1-h^2}{(1-2xh+h^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty}(2n+1)h^n P_n(x). \qquad [7]$$

QUESTION B2

a. By any method find the circulation of the field $\mathbf{F} = (x-y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$ [8]

b. Consider the vector field

$$\boldsymbol{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$$

i. Show that $\mathbf{F}$ is a conservative force field [4]

ii. Find the scalar potential ϕ for the vector field $\mathbf{F} = \nabla\phi$. [5]

iii. Hence, or otherwise, find the work done in moving a particle from $(1,1,-2)$ to $(5,4,10)$ in the force field $\mathbf{F}$. [3]

QUESTION B3

a. Find the work done in moving a particle in the force field

$$\mathbf{F} = (3x^2 - 3x)\mathbf{i} + 3z\mathbf{j} + \mathbf{k}$$

from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve $x = t,\ y = t,\ z = 2t,\quad 0 \leqslant t \leqslant 2.$ [7]

b. Use Green's Theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

to evaluate $\oint_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$ along the curve C, which is a boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. [13]

QUESTION B4

a. Given the vectors $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ and $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$ prove that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ [8]

b. Assume that $\phi(x, y, z)$ is continuously differentable show that $\nabla \times (\nabla\phi) = \mathbf{0}$ [8]

c. Given that $\phi = 4x^2y^3z^7$ find $\nabla \cdot \nabla\phi$ [4]

QUESTION B5

a. Consider the formula

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

for the area of a region bounded by the closed curve C. Evaluate the area when C is the circle centred at the origin and with radius 2. Parametrize C first. [8]

b. Consider the cylindrical polar coordinate system defined by

$$x = \rho\cos\phi,\quad y = \rho\sin\phi,\ z = z.$$

a. Derive the scale factors h_ρ, h_ϕ and h_z. [4]

b. Find the unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ and use them to show that cylindrical coordinate system is orthogonal [8]

END OF EXAMINATION PAPER