University of Swaziland

Main Examination, 2016/2017

BASS III, B.Ed (Sec.) III, B.Sc. III, B.Eng. III

Title of Paper : Complex Analysis

Course Number : M313

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

a) Consider the complex number $\phi = -1 + i$. Determine the following:

   i) Complex conjugate of ϕ. [1]

   ii) Modulus of ϕ. [1]

   iii) $Im(\phi - \bar{\phi})$ [1]

   iv) Multiplicative inverse of ϕ. [2]

   v) Principal value of the argument of ϕ. [2]

b) Show that $|z_1 z_2| = |z_1||z_2|$, where z_1 and z_2 are complex numbers. [5]

c) Express $w = z^2 - 3z - 3i$ in the form $w = u(x, y) + iv(x, y)$. [3]

d) Show that $Re(iv) = -Im(v)$, where $v = x + iy$ and $x, y \in \mathbb{R}$. [3]

e) Find all values of $\rho = (-1 - i)^{1/2}$. [4]

f) Show that $\log(e) = 1 + 2n\pi i$ [3]

g) Determine if $f(z) = 3x^2 \sin(y) - i(3y - x)$ is analytic at $(0, \pi)$. [4]

h) Let C be a positively oriented circle such that $|z| = 1$. Use Cauchy-Goursat theorem to determine

$$\int_C \frac{dz}{z^2 + 2z + 2}$$

[2]

i) Using the known Maclaurin series for $f(z) = \sin(z)$, find the Maclaurin series of

$$f(z) = \sin(z^3).$$

[3]

j) Find the residue of $f(z) = \frac{z-4}{z^2+1}$ at $z = -i$. [3]

k) Determine the order of each pole of

$$f(z) = \left(\frac{z}{2z + 1}\right)^3$$

and the corresponding residue. [3]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

a) Show that

$$\log(-1+i\sqrt{3}) = \ln(2) + 2\pi i\left(n+\frac{1}{3}\right)$$

[7]

b) Let p, and q be complex numbers. Prove that if $pq = 0$, then at least one of the two factors is zero. [6]

c) Let v and w be complex numbers. Show that $|v+w| \geq ||v| - |w||$. [7]

QUESTION B3 [*20 Marks*]

a) Suppose that

$$f(z) = u(x,y) + iv(x,y)$$

and $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Show that the first order partial derivatives of u and v must exist at a point (x_0, y_0) and they must satisfy the Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

[15]

b) Show that the function

$$f(z) = 3x + y + i(3y - x)$$

is entire. [5]

QUESTION B4 [*20 Marks*]

a) Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then prove that

$$\int_C \frac{f(z)dz}{z - z_0} = 2\pi i f(z_0).$$

[15]

b) Show that if C is a positively oriented circle such that $|z| = 2$, then

$$\int_C \frac{z dz}{(9-z^2)(z+i)} = \frac{\pi}{5}.$$

[5]

QUESTION B5 [*20 Marks*]

a) Find the Laurent series that represents the function

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

in the domain $1 < |z| < 2$. [8]

b) Using the fact that $\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$, derive the Maclaurin series for the entire function

$$f(z) = \cos(z).$$

[8]

c) Using the fact that $\cosh(z) = \cos(iz)$ and your answer in part b), find the Maclaurin series for the entire function

$$f(z) = \cosh(z).$$

[4]

QUESTION B6 [*20 Marks*]

a) Let C be a positively oriented circle such that $|z| = 4$. Use Cauchy's residue theorem to show that

$$\int_C \frac{(5z-2)dz}{z(z-1)} = 10\pi i.$$

[7]

b) Suppose that

$$f(z) = \frac{(\ln(r) + i\theta)^3}{z^2+1}, \quad r > 0, \quad 0 < \theta < 2\pi.$$

Find the residue of f at $z = i$. [7]

c) Let C be a positively oriented circle such that $|z| = 1$. Find

$$\int_C \frac{(z-1)}{(z+4)(z-7)} dz$$

[6]

END OF EXAMINATION PAPER