UNIVERSITY OF SWAZILAND

FINAL EXAMINATION, 2016/2017

B.Sc. III, B.Ed III, BASS III

Title of Paper : Abstract Algebra I

Course Number : M323

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked**.
2. Show all your working.

Special Requirements: None

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: Answer ALL Questions

A1. (a) Show that, if $a \equiv b(\mod n)$ and $c \equiv d(\mod n)$, then $ac \equiv bd(\mod n)$ (3)

(b) Prove that, in any group G, the identity element is unique. (3)

(c) The table on the last page may be completed to define a binary operation $*$ on the set $G = \{e, a, b, c\}$ in such a way that $(G, *)$ becomes a group. Assume this is possible and compute the missing entries. (6)

(d) Let H be a proper subgroup of a group G where $|G| = p^2, p$ prime. Show that H is cyclic. (8)

A2. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 1 & 7 & 5 & 3 & 4 & 2 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 8 & 3 & 4 & 5 & 2 & 6 \end{pmatrix}$$

(a) Express α and β as products of disjoint cycles, and then as products of transpositions. For each of them, say whether it is an even permutation or an odd one,

(b) Calculate $\alpha^{-1}, \beta^{-1}\alpha, (\alpha\beta)^{-1}$

(c) Solve the equations $x\alpha = \beta, \alpha y = \beta$

(d) Find the order of β and compute β^{2014} (20)

SECTION B: Answer any *THREE* Questions

QUESTION B1 [*20 Marks*]

B1. (a) Determine whether the set $\mathbb{Z}$ with respect to the binary operation

$$a * b = a + b - 2017$$

is a group (7)

(b) Find the order of $\frac{1}{2}(-1 - i\sqrt{3})$ in $\mathbb{C}^0$, the multiplicative group of complex numbers. (3)

(c) (i) State Lagrange's Theorem (2)

(ii) Prove Lagrange's Theorem (8)

QUESTION B2 [*20 Marks*]

B2. (a) Find the last (unit's) digit of the number 7^{2017} (5)

(b) Let $H =< 3 >$ be the subgroup of $\mathbb{Z}_{15}$.

(i) Find all cosets of H in $\mathbb{Z}_{15}$. (5)

(ii) Write the Cayley table for the group operation in the factor group $\mathbb{Z}_{15}/H$. (3)

(c) Prove that every subgroup of a cyclic group is cyclic. (7)

QUESTION B3 [*20 Marks*]

B3. (a) Let H be the subset $\{(1), (12), (34), (12)(34)\}$ of the group S_4.

(i) Show that H is a subgroup of S_4. (4)

(ii) Show that H is not cyclic (4)

(b) Let : $(G^*) \to (H, 0)$ be an isomorphism of groups.

(i) Prove that, it e_G is the identity element of G, then $\phi(e_G)$ is the identity element of H. (3)

(ii) Prove that, for any $g \in G, (g^{-1})\phi = (g)\phi^{-1}$ (3)

(iii) Prove that $Ker\phi$ is a normal subgroup of G. (6)

QUESTION B4 [*20 Marks*]

B4. (a) (i) Define the term a subgroup of a group. (2)

(ii) Let H be a subgroup of a group G, and let e_H and e_G be the identity elements of H and G respectively. Prove that $e_H = e_G$. (5)

(b) Show that the group $\mathbb{R}$ under addition is isomorphic to $\mathbb{R}^+$ under multiplication. (7)

(c) Prove that if G is finite group of a prime order then G is cyclic. (6)

QUESTION B5 [*20 Marks*]

B5. (a) Show that a finite group of prime order has no proper subgroups (5)

(b) Find the greatest common divisor d of the numbers 102 and 42 and express it in the form $d = 102m + 42n$ for some $m, n, \in \mathbb{Z}$ (4)

(c) Let G be the set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix},$$

where $a, b, c \in \mathbb{Q}, ac \neq 0$.

(i) Show that, with respect to matrix multiplication, G is a group. (5)

(ii) Let $\mathbb{Q}^0$ be the multiplication group of non zero rational numbers.

Show that the mapping $\phi : G \to \mathbb{Q}^0$ by

$$\phi\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right) = \frac{a}{c}$$

is a group hormomorphism. (3)

(iii) Find the kernel of ϕ. (3)

END OF EXAMINATION PAPER

⋆	e	a	b	c
e	e	a	b	c
a	a	e		b
b	b		a	
c	c	b		a