University of Swaziland

Supplementary Examination, 2016/2017

B.Sc. III, B.Ed III, BASS III

Title of Paper : Abstract Algebra I

Course Number : M323

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked**.
2. Show all your working.

Special Requirements: None

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: Answer ALL Questions

A1. (a) Find all subgroups of $\mathbb{Z}_{18}$ and draw the lattice diagram (10)

(b) Let G and H be groups, $\varphi : G \to H$ be an insomorphism of G and H and let e be the identity of G. Prove that $(e)\varphi$ is the identity in H and that $(a^{-1})\varphi = [(a)\varphi]^{-1}$ for all $a \in G$. (10)

A2. (a) For each binary operation $*$ defined on a set G, say whether or not $*$ gives a group structure on the set

(i) Define $*$ on $\mathbb{Q}^+$ by $a * b = \frac{ab}{2} \quad \forall a, b \in \mathbb{Q}^+$ (5)

(ii) Define $*$ on $\mathbb{R}$ by $a * b = ab + a + b \quad \forall a, b \in \mathbb{R}$ (5)

(b) Show that $\mathbb{Z}_6$ and S_3 are NOT isomorphic and that $\mathbb{Z}$ and $5\mathbb{Z}$ are isomorphic. (10)

SECTION B: Answer any *THREE* Questions

QUESTION B1 [*20 Marks*]

B1. (a) (i) Define the notion of "NORMAL SUBGROUP" of a group (4)

(ii) Verify that the subgroup $H = \{(1), (123), (132)\}$ is a normal subgroup of S_3 (6)

(b) Prove that every subgroup of a cyclic group is cyclic. (10)

QUESTION B2 [*20 Marks*]

B2. (a) Prove that a non-abelian group of order $2p, p$ prime, contains at least one element of order p. (6)

(b) Consider the following permutations in S_6

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

Compute (i) $P\sigma$ (ii) σ^2 (iii) σ^{-1} (iv) σ^{-2} (v) σ^2 (10)

(c) Write the permutations in (b) as a product of disjoint cycles in S_6 (4)

QUESTION B3 [*20 Marks*]

B3. (a) Suppose that m, a, b are positive integers such that $(a, m) = 1$ and $(b, m) = 1$ Prove that $(ab, m) = 1$ (10)

(b) (i) Express $d = (2190, 465)$ as an integral linear combination of 2190 and 465 (5)

(ii) Solve the following $3x \equiv 5(\mod 11)$ (5)

QUESTION B4 [*20 Marks*]

B4. (a) (i) State Cayley's theorem (4)

(ii) Let $(\mathbb{R}^+, \cdot)$ be the multiplicative group of all positive real numbers and $(\mathbb{R}, +)$ be the additive group of all real numbers. Show that $(\mathbb{R}^+, \cdot)$ is isomorphic to $(\mathbb{R}, +)$ (6)

(b) (i) Find the number of generators in each of the following cyclic groups $\mathbb{Z}_{30}$ and $\mathbb{Z}_{42}$ (5)

(ii) Determine the right cosets of $H =< 4 >$ in $\mathbb{Z}_8$ (5)

QUESTION B5 [*20 Marks*]

B5. (a) Show that $\mathbb{Z}_p$ has no proper subgroup if p is prime (6)

(b) Show that if $(a, m) = 1$ and $(b, m) = 1$ then $(ab, m) = 1$ $a, b, m \in \mathbb{Z}$ (6)

(c) Prove that every group of prime order is cyclic (8)

END OF EXAMINATION PAPER