UNIVERSITY OF SWAZILAND

EXAMINATION, 2016/2017

BASS IV, B.Ed (Sec.) IV, B.Sc IV

Title of Paper : NUMERICAL ANALYSIS II

Course Number : M411

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) What values should the parameters a and b have so that the following multi-step method is consistent? [4 marks]

$$y_{i+2} - ay_{i+1} - 2y_i = bhf_i$$

(b) Use the Taylor series expansions and the method of undetermined coefficients to derive the two-step Adams-Bashforth (Explicit) method. [6 marks]

(c) A numerical scheme has been used to approximate the solution of

$$\frac{dy}{dt} = t^2 - y^2$$

and has given the following estimates, to 6 decimal places

$$y(0.3) \approx 1.471433, \quad y(0.32) \approx 1.447892$$

Now use the 2-step, explicit linear multistep scheme

$$y_{n+2} - 1.6y_{n+1} + 0.6y_n = h(5f_{n+1} - 4.6f_n)$$

to approximate $y(0.34)$. [5 Marks]

(d) Find the Lipschitz constant for $f(t, y)$ in the initial value problem [3 marks]

$$y' = -(1+t^2)y + \sin t, \quad y(0) = 1, \quad 0 \le t \le 1.$$

(e) Prove that the constant function $y(x) = a$ that fits inconsistent measurements $y_1, y_2, \ldots, y_n$ in the least-squares sense corresponds to the mean value [4 marks]

$$a = \frac{1}{n}\sum_{k=1}^{n} y_k$$

(f) Use the Gram-Schmidt procedure to construct ϕ_1, ϕ_2, ϕ_3 where $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ is an orthogonal set on $[-1, 1]$ with respect to the weight function $w(x) = 1$, given that $\phi_0(x) = 1$. [5 marks]

(g) Write down an $0(h^2)$ finite difference scheme for the following boundary value problem:

$$-\frac{d^2u}{dx^2} + c(x)u = f(x), \quad 0 \le x \le 1.$$
$$u(0) = \alpha, \quad u(1) = \beta$$

where $c(x) \ge 0$, $f(x)$ are given continuous functions in the interval $[0, 1]$ and α and β are known boundary values of $u(x)$. [4 marks]

(h) Consider the following first order PDEs $\quad \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0$, $(a > 0)$

i. Derive the finite difference scheme based on forward difference in time and backward difference in space. [4 marks]

ii. What is the order of accuracy for this scheme? [5 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Suppose that $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is an orthogonal set of functions on an interval $[a, b]$ with respect to the weight function $w(x)$ and the least squares approximation to $f(x)$ on $[a, b]$ with respect to $w(x)$ is

$$P_n(x) = \sum_{j=0}^{n} \tau_j \phi_j(x)$$

where, for each $j = 0, 1, \ldots, n$. By minimising the least squares error,

$$E(\tau_0, \tau_1, \ldots, \tau_n) = \int_a^b w(x) \left[f(x) - \sum_{k=0}^{n} \tau_k \phi_k(x) \right]^2 dx$$

show that

$$\tau_j = \frac{\int_a^b w(x)\phi_j(x)f(x)\, dx}{\int_a^b w(x)[\phi_j(x)]^2} = \frac{1}{\alpha_j} \int_a^b w(x)\phi_j(x)f(x)\, dx$$

where [7 marks]

$$\alpha_j = \int_a^b w(x)[\phi_j(x)]^2$$

(b) Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 1\,, & -\pi < x < 0 \\ 0\,, & 0 < x < \pi. \end{cases}$$

i. Show that the least squares trigonometric polynomial that approximates $f(x)$ in the interval $-\pi < x < \pi$ is [10 marks]

$$\begin{aligned} S_n(x) &= \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{k\pi} \left[(-1)^k - 1 \right] \sin kx \\ &= \frac{1}{2} - \frac{2}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \ldots + \frac{1}{(2n-1)} \sin[(2n-1)x] \right] \end{aligned}$$

for $n = 1, 2, \ldots$

ii. By giving an appropriate value to x, show that as $n \to \infty$. [3 marks]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

QUESTION B3 [*20 Marks*]

B3 (a) A multi-step method for solving the initial value problem (IVP) [6 marks]

$$y'(x) = f(x, y),\ a \le x \le b,\ y(a) = \alpha$$

is defined by the difference equation

$$y_{n+2} = -\,3y_n + 4y_{n+1} - 2hf(x_n, y_n);\ n = 0, 1, \ldots, N-2$$

with starting values y_0 and y_1. Use this method to solve

$$y'(x) = 2 - y,\ 0 \le x \le 1,\ y(0) = 0$$

for $y(0.2)$ and $y(0.3)$ with $h = 0.1$, and starting values $y_0 = 0$ and $y_1 = 2 - e^{-0.1}$.

(b) Derive the following 3-step Adams-Bashforth explicit method

$$y_{i+1} = y_i + \frac{h}{12}\left[23f(t_i, y_i) - 16f(t_{i-1}, y_{i-1}) + 5f(t_{i-2}, y_{i-2})\right],$$

with given values of y_0, y_1, y_2 for solving the IVP $y'(t) = f(t, y(t))$. [7 marks]

(c) Analyze the consistency, zero-stability and convergence of the 3-Step Adams-Bashforth method. [7 marks]

QUESTION B4 [*20 Marks*]

B4 (a) The Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12xy, \qquad 0 < x < 1.5, \qquad 0 < y < 1,$$

is to be solved subject to

$$\begin{aligned} u(x,0) &= 0, \qquad u(x,1) = 6x, \qquad 0 < x < 1.5; \\ u(0,y) &= 0, \qquad u(1.5,y) = 3y^2, \qquad 0 < y < 1. \end{aligned}$$

Use finite differences with $h = k = 0.5$ to find an approximate solution to this equation. [10 Marks]

(b) By replacing y' and y'' using central difference schemes, write the general discretization on 5 sub-intervals of the following boundary value problem

$$y'' = xy' - 3y + e^x, \quad 0 \le x \le 1$$

$$y(0) = 1, \quad y(1) = 2$$

in matrix-vector form $\mathbf{Aw} = \mathbf{b}$ (*Do NOT solve!!!*) [10 marks]

QUESTION B5 [*20 Marks*]

B5 (a) Derive the Taylor method of order two

$$y_{i+1} = y_i + h\left[f(t_i, y_i) + \frac{h}{2}f'(t_i, y_i)\right]$$

for solving the initial-value problem $y'(t) = f(t, y(t)), \ y(a) = \alpha$. Assume that $y(t) \in C^{(3)}[a, b]$. [6 marks]

(b) Use the Taylor method of order two solve the initial-value problem

$$y' = \frac{1+t}{1+y}, \ 1 < t < 2, \ y(1) = 2, \text{ with } h = 0.5$$

[6 marks]

(c) Use the 4th-order Runge-Kutta method to solve the initial-value problem

$$y' = 1 + (t - y)^2, \ 2 < t < 3, \ y(2) = 1, \text{ with } h = 0.5$$

[8 marks]

QUESTION B6 [*20 Marks*]

B6 (a) Prove that the quadratic least squares approximation to $f(x) = e^x$ on $[-1, 1]$ is

$$P_2(x) = \frac{15\left(e^2 - 7\right)x^2}{4e} + \frac{3x}{e} - \frac{3\left(e^2 - 11\right)}{4e}$$

[7 Marks]

(b) Find an equation of the form $f(x) = ae^{x^2} + bx^3$ that approximates the data in the table

x	- 1	0	1
y	0	1	2

in the least squares sense. [9 Marks]

(c) Derive the *explicit* finite finite difference scheme for the heat equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

[4 Marks]

END OF EXAMINATION PAPER