UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2016/2017

BASS IV, B.Ed (Sec.) IV, B.Sc IV

Title of Paper : NUMERICAL ANALYSIS II

Course Number : M411

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Find the quadratic least squares approximation of the function

$$f(x) = x^4 + x + 2$$

on the interval $[-1, 1]$ [6 marks]

(b) Use the Gram-Schmidt process to construct orthogonal polynomials

$$\phi_1(x), \phi_2(x)$$

on the interval $[0, 1]$ [4 marks]

(c) Derive the recurrence formula

$$T_0(x) = 1, \;\; T_1(x) = 1, \;\; T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

where T_n are Chebyshev polynomials of order n defined by [4 marks]

$$T_n(x) = \cos(n \arccos(x)), \;\; \text{for each } \; n \geq 0 \; \text{ with } \; x \in [-1, 1]$$

(d) Show that $f(t, y) = t|y|$ satisfies the Lipschitz condition on [4 marks]

$$D = \{(t, y)| \;\; 1 \leq t \leq 2, \;\; -3 \leq y \leq 4\}$$

(e) Consider the following multi-step method for approximating the solution of an initial value problem,

$$y_{i+1} = 2y_i - y_{i-1} + \frac{h}{4}[f_{i-2} + 3f_{i-1}],$$
$$y_0 = a, \;\; y_1 = a_1, \;\; y_2 = a_2.$$

Discuss the stability, consistency and convergence of this method. [7 marks]

(f) A numerical scheme has been used to approximate the solution of

$$\frac{dy}{dt} = t + y, \qquad y(0) = 3$$

and has produced the following estimates, to 6 decimal places

$$y(0.4) = 4.509822, \quad y(0.45) = 4.755313$$

Now use the 2-step, explicit linear multistep scheme

$$y_{i+2} - y_{i+1} = h(1.5f_{i+1} - 0.5f_i)$$

to approximate $y(0.5)$. [4 marks]

(g) Derive the *explicit* finite finite difference scheme for the heat equation. [4 marks]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(h) Consider the finite difference scheme for solving a parabolic partial differential equation,

$$u_j^{n-1} = -\alpha u_{j-1}^n + (1 + 2\alpha)u_j^n - \alpha u_{j+1}^n.$$

Show that the scheme is unconditionally stable by performing a Von-Neumann analysis. [7 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

1. (a) Suppose $f \in C[a,b]$ and that a polynomial $P_n(x) = \sum_{k=0}^{n} \tau_k x^k$ is required to minimize the error

$$\int_a^b [f(x) - P_n(x)]^2 \; dx.$$

Derive the normal equations that can be used used to evaluate the coefficients τ_k ($k = 0, 1, 2, \ldots,$) [5 marks]

(b) Prove that the quadratic least squares approximation to $f(x) = e^x$ on $[-1, 1]$ is [7 marks]

$$P_2(x) = \frac{15\,(e^2 - 7)\,x^2}{4e} + \frac{3x}{e} - \frac{3\,(e^2 - 11)}{4e}$$

(c) Prove the orthogonality property of Chebyshev polynomials with respect to the weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$.

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{2}, & \text{if } m = n \end{cases}$$

[8 marks]

QUESTION B3 [*20 Marks*]

B3 (a) The 4th order Milne's method for solving initial value problems is given by

$$y_{i+1} = y_{i-3} + \frac{4h}{3}\left[2f_i - f_{i-1} + 2f_{i-2}\right].$$

Analyse the stability of this method. [4 marks]

(b) Use any method to derive the following 2-Step Adams-Moulton method [8 marks]

$$\begin{aligned} y_0 =& \alpha, \; y_1 = \alpha_1, \\ y_{i+1} =& y_i + \frac{h}{12}[5f(t_{i+1}, y_{i+1}) + 8f(t_i, y_i) - f(t_{i-1}, y_{i-1})], \quad i = 1, 2, \ldots, N-1 \end{aligned}$$

(c) Use the Runge-Kutta method of order 4 with $h = 0.1$ to approximate the solution of the following initial value problem at $t = 2.2$

$$y'(t) = 1 + (t-y)^2, \quad 2 \leq t \leq 3, \quad y(2) = 1$$

[8 marks]

QUESTION B4 [*20 Marks*]

B4 (a) Consider the elliptic partial differential equation

$$\begin{aligned} u_{xx} + u_{yy} =& 0, \quad 0 \leq x \leq 2, \; 0 \leq y \leq 3, \\ u(x,0) =& x/2, \; u(x,3) = 1, \; 0 \leq x \leq 2, \\ u(0,y) =& y/3, \; u(2,y) = 1, \; 0 \leq y \leq 3. \end{aligned}$$

Use finite differences on a uniform grid, with $h = k = 1$, to approximate both $u(1,1)$ and $u(1,2)$. [10 marks]

(b) Write down an $\mathcal{O}(h^2)$ finite difference scheme for the following boundary value problems and express the finite difference scheme in matrix form $\mathbf{AU} = \mathbf{F}$

$$u'' + 2xu' + 3u - \sin(x) = 0, \quad a \le x \le b.$$
$$u'(a) + u(a) = \alpha, \quad u'(b) + u(b) = \beta$$

where $p(x)$, $q(x)$ and $f(x)$ are given continuous functions in the interval $[a,b]$ and α and β are given constant values. [10 marks]

QUESTION B5 [*20 Marks*]

B5 (a) Let $f(x)$ be a function defined as

$$f(x) = \begin{cases} x + \pi\ , & -\pi < x < 0 \\ \pi - x\ , & 0 < x < \pi. \end{cases}$$

i. Show that the least squares trigonometric polynomial that approximates $f(x)$ in the interval $-\pi < x < \pi$ is

$$\begin{aligned} S_n(x) &= \frac{\pi}{2} - \sum_{k=1}^{n} \frac{2\left[(-1)^k - 1\right]}{\pi k^2} \cos kx \\ &= \frac{\pi}{2} + \frac{4}{\pi}\left[1 + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \frac{\cos 9x}{9^2} + \dots\right] \end{aligned}$$

for $n = 1, 2, \dots$ [9 marks]

ii. By giving an appropriate value to x, show that as $n \to \infty$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$$

[4 marks]

(b) If an $O(k^2 + h^2)$ numerical method for solving the heat equation is constructed using the central difference quotient
to approximate u_t and the usual difference quotient to approximate u_{xx},
show that the resulting difference problem is

$$u_{i,j+1} = u_{i,j-1} + 2\lambda(u_{i+1,j} - u_{i,j} + u_{i-1,j}),$$

defining λ appropriately. [7 marks]

QUESTION B6 [*20 Marks*]

B6 (a) Use Euler's method with $h = 0.1$ to approximate the solution of the initial value problem

$$y''' + 3y'' + 3y' + y = -4\sin t, \quad y(0) = 1,\ y'(0) = 1,\ y''(0) = -1$$

at $t = 0.2$. [6 Marks]

(b) The Legendre polynomials are given by $\phi_0(x) = 1$, $\phi_1(x) = x$ and satisfy the recurrence relation

$$\phi_{n+1}(x) = \frac{2n+1}{n+1}x\phi_n(x) - \frac{n}{n+1}\phi_{n-1}(x), \quad \text{for } n \geq 1.$$

i. Use the recurrence relation to find $\phi_2(x), \phi_3(x)$ and $\phi_4(x)$. [2 marks]

ii. The first three Legendre polynomials are defined on $[-1, 1]$ as

$$\phi_0(x) = 1, \quad \phi_1(x) = x, \quad \phi_2(x) = x^2 - \frac{1}{3}$$

Use Legendre polynomials of degree $n = 2$ with weight $w(x) = 1$ to approximate $f(x) = |x|$ on the interval $[-1, 1]$. [6 marks]

(c) Find the continuous least-squares trigonometric polynomial $S_2(x)$ for $f(x) = x^2$ on $[-\pi, \pi]$. [6 marks]

END OF EXAMINATION PAPER