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University of Swaziland

Final Examination, December 2016

B.A.S.S. , B.Sc, B.Eng, B.Ed

Title of Paper : Introduction to Partial Differential Equations

Course Number : M415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a). **SECTION A(COMPULSORY): 40 MARKS**
   Answer ALL QUESTIONS.
   b). **SECTION B: 60 MARKS**
   Answer ANY THREE questions.
   Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Non programmable calculators may be used (unless otherwise stated).
5. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

Question 1

(a) Consider the partial differential equation

$$u_{xx} - 2\sin(x)u_{xy} - \cos^2(x)u_{yy} - \cos(x)u_y + e^x = 0.$$

(i) Classify the partial differential equation by stating its order, linearity, homogeneity, and kind of coefficients.. [2]

(ii) Determine whether the given partial differential equation is hyperbolic, parabolic or elliptic. [2]

(iii) Determine the characteristic curves $\xi(x,y)$ and $\eta(x,y)$. [6]

(b) Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$:

$$p_{tt} - 4p_{xx} = 0, \quad 2p(x,0) - \sin(x) = 0, \quad p_t(x,0) = \frac{4x}{3}.$$

Determine $p\left(\frac{\pi}{2}, 3\pi\right)$. [5]

(c) Using Laplace transform, determine $v(x,t)$.

$$v_{xt} + \sin(t) = 0, \quad v_x(x,0) = x, \quad v(0,t) = 0.$$

[6]

(d) Derive Parseval's identity theorem for the summability of the Fourier series coefficients of a function. [5]

(e) Consider the wave equation

$$\begin{aligned}
\phi_{tt} &= c^2\phi_{xx}, \quad 0 < x < \pi, \quad t > 0, \\
\phi(x,0) &= 3\sin(x), \quad 0 \le x \le \pi, \\
\phi_t(x,0) &= 0, \\
\phi(0,t) &= \phi(\pi,t) = 0.
\end{aligned}$$

Write down the ordinary boundary value problems for $X(x)$ and $T(t)$ that must be solved in order to obtain the solution of the wave equation using the method of separation of variables. [5]

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(f) Suppose that the temperature distribution in a rod of length π is given by $T(x,t)$. We assume that one end is kept at zero temperature and the other end ($x = \pi$) is insulated such that there is no heat flow. Write down a model that could be used to determine the temperature distribution $T(x,t)$, provided that the initial temperature distribution is given by $x^2 \cosh(\pi x)$. [3]

(g) By eliminating the arbitrary function, find the partial differential equation satisfied by

$$u(x,y) = x + y + f(xy).$$

[3]

(h) Given that the arbitrary function is differentiable, show that

$$u(x,y) = e^{-x/4} f(3x - 4y)$$

satisfies

$$4u_x + 3u_y + u = 0.$$

[3]

SECTION B: ANSWER ANY 3 QUESTIONS

Question 2

Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$:

$$\begin{aligned} p_{tt} &= v^2 p_{xx}, \\ p(x,0) &= \phi(x), \\ p_t(x,0) &= \psi(x), \end{aligned}$$

where v is a constant. Show that the solution of the wave equation is given by:

$$p(x,t) = \frac{1}{2}\left(\phi(x+vt) + \phi(x-vt) + \frac{1}{v}\int_{x-vt}^{x+vt} \psi(\gamma)d\gamma\right)$$

[20]

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Question 3

Consider the heat equation

$$
\begin{aligned}
u_t &= 4u_{xx}, \qquad 0 < x < 4, \qquad t > 0,\\
u(x,0) &= 4, \qquad 0 \le x \le 4,\\
u(0,t) &= \pi,\\
u(4,t) &= 3e.
\end{aligned}
$$

Determine $u(x,t)$ using the method of separation of variables. [20]

Question 4

Consider the radioactive decay problem given by

$$
\begin{aligned}
u_t &= ku_{xx} + Ae^{-\alpha x}, \qquad 0 < x < \pi, \qquad t > 0,\\
u(x,0) &= \sin(x), \qquad 0 \le x \le \pi,\\
u(0,t) &= 0,\\
u(\pi,t) &= 0,
\end{aligned}
$$

where A and α are constants. Find $u(x,t)$ using the method of separation of variables. [20]

Question 5

(a) Consider Laplace's equation in a circle with Dirichlet boundary conditions.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi} = 0, \qquad 0 < r < 2, \qquad -\pi < \phi < \pi,$$
$$u(2,\phi) = \phi, \qquad -\pi < \phi < \pi.$$

Use the method of separation of variables to determine $u(r,\phi)$. [17]

(b) Consider the Dirichlet problem of a sphere

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) = 0,$$
$$u(e,\phi) = \phi^2 - 3\cos(\phi), \qquad 0 \le \phi \le \pi.$$

What is the radius of the sphere? [3]

Question 6

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(a) Solve the first order partial differential equation

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

subject to $u = \cos(x)$ on $x^2 + y^2 = 1$. [7]

(b) Use Laplace transforms to find a solution

$$\begin{aligned} u_{xx} - \frac{1}{c^2}u_{tt} + k\sin(\pi x) &= 0, \quad 0 < x < 1, \quad t > 0, \\ u(x,0) &= 0, \quad 0 \leq x \leq 4, \\ u_t(x,0) &= 0, \\ u(0,t) &= 0, \\ u(0,t) &= 0 \end{aligned}$$

[8]

(c) Solve the first order partial differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 1$$

[5]