University of Swaziland

158

Final Examination, 2016/2017

B.Sc. IV, BASS IV, BED. IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked.**
2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

A1. (a) Let F be a field. Explain what is meant by saying "The polynomial $f(x)$ is irreducible in F"' (2)

(b) State Eisenstein's irreducibility criterion (3)

(c) Determine the irreducibility or otherwise of

(i) $x^3 - 7x^2 + 3x + 3 \in \mathbb{Q}[x]$ (5)

(ii) $2x^{10} - 25x^3 + 10x^2 - 30 \in \mathbb{Q}[x]$ (5)

(d) Suppose

$$f(x) = x^5 + 5x^4 + 3x + 2 \quad g(x) = 2x^2 + 1$$

are polynomials $\mathbb{Z}_7(x)$. Find $q(x)$ and $\nu(x)$ in $\mathbb{Z}_7$ as described in the division algorithm so that

$$f(x) = q(x)g(x) + \nu(x)$$

with $\nu(x) = 0$ or degree $\nu(x) <$ degree $g(x)$ (5)

A2. (a) Let α be an element of an extension field(s) of F. Explain what is meant for α to be

(i) Algebraic over F (3)

(ii) Transcendental over F (2)

(b) Consider the polynomial $x^3 + x^2 + 1$

(i) Show that $x^3 + x^2 + 1$ is irreducible over $\mathbb{Z}_2$ (3)

(ii) Let α be a zero of $x^3 + x^2 + 1$ in the extension field of $\mathbb{Z}_2$. Show that $x^3 + x^2 + 1$ factors into three linear factors in $\mathbb{Z}_2(\alpha)$ by finding this factorization. (12)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

(a) Use Fermat's theorem to complete the remainder when 8^{123} is divided by 13 (6)

(b) For each of the following, find irred (α, θ) and deg(α, θ)

(i) $\sqrt{3} + i$ (7)

(ii) $\sqrt{\frac{1}{5} + \sqrt{7}}$ (6)

(c) Show that if a polynomial $f(x) \in Z(x)$ is reducible over $\mathbb{Q}$ then its also reducible over $\mathbb{Z}$ (8)

QUESTION B4 [*20 Marks*]

(a) Show that for a field F, the set of all matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} \quad a_{ij} \in F$$

is a right ideal but not a left ideal of $M_2(F)$ (6)

(b) Let $\varphi_\alpha : \mathbb{Z}_7(x) \to \mathbb{Z}_7$. Evaluate each of the following for the indicated evaluation homomorphism

(i) $\varphi_2(3x^{79} + 5x^{53} + 2x^{43})$

(ii) $\varphi_3[(x^3+2)(4x^2+3)(x^7+3x^2+1)]$ (10)

(c) Show that if D is an integral domain, then $D[x]$ is also an integral domain (4)

QUESTION B5 [*20 Marks*]

(a) Prove that every field is an integral domain (6)

(b) Which of the following are rings with the usual addition and multiplication

(i) $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$

(ii) $\{ri : r \in \mathbb{R}, \quad i = -1\}$ (8)

(c) Mark each of the following true or false

(i) Every finite integral domain is a field.

(ii) Every division ring is commutative.

(iii) $\mathbb{Z}_6$ is not an integral domain (6)

QUESTION B6 [*20 Marks*]

(a) Classify each of the given $\alpha \in \vartheta$ as algebraic or transectional over the given fieLd F. If α is algebraic over F find $\deg(\alpha, F)$

(i) $\alpha = 1 + i, F = \mathbb{Q}$

(ii) $\alpha = \sqrt{\pi}, F = Q[\pi]$

(iii) $\alpha = \pi^2, F = \mathbb{Q}$

(iv) $\alpha = \pi^2, F' = Q(\pi^2)$

(v) $\alpha = \pi^2, F = Q(\pi)$ (10)

(b) Show that the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ is NOT a field. (5)

(c) Find a polynomial of degree > 0 in $\mathbb{Z}_4(x)$ that is a unit. (5)

QUESTION B7 [*20 Marks*]

(a) Suppose F is a field f is an irreducible polynomial over F and f, h are polynomials over F such that f divides gh. Show that either f divides g or f divides h (10)

(b) Define an ideal N of a ring R (2)

(c) Find all ideals of $\mathbb{Z}_{10}$ and all maximal ideals of $\mathbb{Z}_{18}$ (8)

END OF EXAMINATION PAPER