UNIVERSITY OF SWAZILAND 165

EXAMINATION, 2016/2017

BASS IV, B.Ed (Sec.) IV, B.Sc. IV

Title of Paper : Metric Spaces

Course Number : M431

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth a total of 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, A2, B3 – B7) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

QUESTION A1 [20 *Marks*]

(a) i. Give a precise definition of a metric space. (3)

ii. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{otherwise.} \end{cases}$$

Show that d is a metric on $\mathbb{R}^2$. (7)

(b) Let (X, d) be a metric space. Give precise definitions of the following.

i. An open subset U of X. (2)

ii. The distance from $x \in X$ to a subset A of X. (2)

iii. The diameter of a subset A of X. (2)

iv. A convergent sequence $\{x_n\}$ in X. (2)

v. A Cauchy sequence $\{x_n\}$ in X. (2)

QUESTION A2 [20 *Marks*]

(a) Let (X, d) be a metric space.

i. Prove that for $x \in X$, the open ball $B(x, r)$ is an open subset of X. (4)

ii. Prove: *If* $\{U_1, U_2, \ldots, U_n, \ldots\}$ *is any collection of open subsets of* X, *then* $\bigcup_i U_i$ *is open.* (4)

(b) For each metric on $\mathbb{R}^2$ below, describe (or draw a picture of) the open ball $B(0,1)$, where $0 = (0,0)$. (No need to prove that d is indeed a metric.)
For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $\mathbb{R}^2$,

i. $d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$ (3)

ii. $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$. (3)

iii. $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. (3)

iv. $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. (3)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [20 *Marks*]

(a) True or False? (Explain): *Every Cauchy sequence $\{x_n\}$ in a metric space (X, d) converges to some $x \in X$.* (4)

(b) Let (X, d) be a metric space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in X. Suppose $\{y_n\}$ is a Cauchy sequence and $d(x_n, y_n) \to 0$ as $n \to \infty$.

Prove:

i. $\{x_n\}$ is also a Cauchy sequence; (5)

ii. $\{x_n\}$ converges to a limit x in X if and only if $\{y_n\}$ converges to x also. (5)

(c) Prove that every Cauchy sequence $\{x_n\}$ in a metric space (X, d) is bounded. (6)

QUESTION B4 [20 *Marks*]

(a) Let $I = [a, b]$ and let $C(I)$ be the set of all continuous functions on I. For $f, g \in C(I)$, define

$$d(f, g) = \int_a^b |f(x) - g(x)| dx.$$

i. Show that d is a metric on $C(I)$. (8)

ii. Let $I = [0, 1]$ and let $f(x) = x^2$ and $g(x) = x$. Find $d(f, g)$. (4)

(b) For $x, y \in \mathbb{R}$ define d by $d(x, y) = (x - y)^2$. Show that d is not a metric on $\mathbb{R}$. (4)

(c) Let (X, d) be a metric space. Show that for $x, y, z \in X$,

$$|d(x, z) - d(y, z)| \leq d(x, y).$$

(4)

QUESTION B5 [20 *Marks*]

(a) i. Define the interior of a subset A of a metric space X. (2)
ii. What is the interior of $\mathbb{Q}$ in $\mathbb{R}$? Explain your reasoning. (3)

(b) i. Define the boundary of a subset A in a metric space X. (2)
ii. What is the boundary of the set $A = [0,1] \cup \{2\}$ in $\mathbb{R}$? (2)

(c) i. Define a limit point of a subset A of a metric space X. (2)
ii. What are the limit points of the set $A = [0,1] \cup \{2\}$ in $\mathbb{R}$? (3)

(d) Let A° denote the set of interior points of a subset A of a metric space X. Prove that A is open if and only if $A^\circ = A$. (6)

QUESTION B6 [20 *Marks*]

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function.

(a) Give the ε, δ definition for f to be continuous at $x_0 \in X$. (3)

(b) Give the ε, δ definition for f to be uniformly continuous. (3)

(c) Let $a \in X$, $Y = \mathbb{R}$ and $d_Y(x,y) = |x - y|$. Define $f : (X, d_X) \to (Y, d_Y)$ by $f(x) = d_X(x, a)$. Show that f is uniformly continuous over X. (7)

(d) Suppose f is continuous at x_0. Prove that if $\{x_n\}$ is a sequence in X such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$. (7)

QUESTION B7 [20 *Marks*]

(a) What does it mean to say that a metric space (X, d) is complete? (3)

(b) Give an example to show that the set $\mathbb{Q}$ of rational numbers is not complete. (5)

(c) Let (X, d) be a metric space. What does it mean to say that $f : X \to X$ is a contraction mapping on X? (3)

(d) State (do not prove) the contraction mapping theorem. (4)

(e) Show that if $f : X \to X$ is a contraction mapping on a metric space (X, d), then f is uniformly continuous over X. (5)

END OF EXAMINATION PAPER