UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2016/2017

BASS IV, B.Ed (Sec.) IV, B.Sc. IV

Title of Paper : Metric Spaces

Course Number : M431

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth a total of 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, A2, B3 – B7) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

QUESTION A1 [20 *Marks*]

(a) i. Give a precise definition of a metric space. (3)

ii. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

Show that d is a metric on $\mathbb{R}^2$. (5)

(b) Let (X, d) be a metric space. Give precise definitions of the following.

i. A closed ball $\bar{B}(x,r)$ in X. (2)

ii. A closed subset U of X. (2)

iii. The distance from a subset A to a subset B of X. (2)

iv. The diameter of a subset A of X.

v. A limit point of a subset A of X. (2)

vi. The interior of a subset A of X. (2)

vii. The boundary of a subset A of X. (2)

QUESTION A2 [20 *Marks*]

(a) Prove: *If $\{U_1, U_2, \ldots, U_n\}$ is any finite collection of open subsets of X, then $\bigcap_{i=1}^{n} U_i$ is open.* (4)

(b) For each metric on $\mathbb{R}^2$ below, describe (or draw a picture of) the open ball $B(a, 1)$, where $a = (1, 0)$. (No need to prove that d is indeed a metric.)
For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $\mathbb{R}^2$,

i. $d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$ (4)

ii. $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$. (4)

iii. $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. (4)

iv. $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. (4)

END OF SECTION A – TURN OVER

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

(a) Define a Cauchy sequence in a metric space (X, d). (2)

(b) Prove: *Every convergent sequence $\{x_n\}$ in a metric space (X, d) is a Cauchy sequence.* (6)

(c) Let (X, d) be a metric space and define a new metric d' on X by

$$d'(x, y) = \min\{1, d(x, y)\}.$$

Prove that $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if it is a Cauchy sequence in (X, d'). (6)

(c) Prove that every Cauchy sequence $\{x_n\}$ in a metric space (X, d) is bounded. (6)

QUESTION B4 [*20 Marks*]

(a) Let $I = [a, b]$ and let $C(I)$ be the set of all continuous functions on I. For $f, g \in C(I)$, define

$$d(f, g) = \max_{t \in I} |f(t) - g(t)|.$$

i. Show that d is a metric on $C(I)$. (8)

ii. Let $I = [0, 1]$ and let $f(x) = x^2$ and $g(x) = x$. Find $d(f, g)$. (4)

(b) For $x, y \in \mathbb{R}$ define d by $d(x, y) = (x - y)^2$. Show that d is not a metric on $\mathbb{R}$. (4)

(c) Find all limit points of the set $A = [-1, 1] \cup \{3\}$ in $\mathbb{R}$. (4)

QUESTION B5 [20 *Marks*]

(a) Define an open subset U of a metric space (X, d). (2)

(b) Define the closure $\bar{A}$ of a subset of a metric space (X, d). (2)

(c) Define a dense subset A of a metric space X. (2)

(d) Prove that if x is a limit point of a subset A of a metric space (X, d), then every open ball $B(x, r)$ contains an infinite number of elements of A. (7)

(e) Let A° denote the set of interior points of a subset A of a metric space X. Prove that A is open if and only if $A^\circ = A$. (7)

QUESTION B6 [20 *Marks*]

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function.

(a) Give the definition for f continuous at x_0 in terms of balls. (3)

(b) Prove that f is continuous over X if and only if for any open subset G in Y, $f^{-1}(G)$ is open in X. (10)

(c) Suppose f is continuous at x_0. Prove that if $\{x_n\}$ is a sequence in X such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$. (7)

QUESTION B7 [20 *Marks*]

(a) What does it mean to say that a metric space (X, d) is complete? (3)

(b) Let (X, d) be a metric space. What does it mean to say that $f : X \to X$ is a contraction mapping on X? (3)

(c) State (do not prove) the contraction mapping theorem. (4)

(d) Prove: *A subspace A of a complete metric space (X, d) is complete if and only if A is closed in A.* (10)

END OF EXAMINATION PAPER