UNIVERSITY OF SWAZILAND

MAIN EXAMINATION, 2016/2017

BASS IV, B.Ed. (Sec.) IV, B.Sc. IV

Title of Paper : Fluid Dynamics

Course Number : M455

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

a) Consider the three term operator

$$\frac{d}{dt} = \mathbf{q} \cdot \nabla + \frac{\partial}{\partial t}.$$

State what each term of this operator represents and the special names associated with each term of the operator. [6]

b) Define the following terms. Enhance your explanations by drawing a graph showing both the pathlines and a streakline.

i) pathline [2]

ii) streakline [2]

c) State Milne-Thomson Circle theorem. [3]

d) Determine the complex potential w for a doublet at $z = 4$ whose axis makes an angle $\alpha = \frac{\pi}{2}$ with the real axis and a source of strength $\mu = e^{\pi}$. [3]

e) At a point in an incompressible fluid having spherical coordinates (r, θ, ψ), the velocity potential is given by

$$\phi = Ur\cos(\theta)$$

where U is a constant.

i) Find the components of the velocity. [3]

ii) Find the equations of the streamlines. [3]

iii) Confirm that the motion is irrotational. [3]

iv) Confirm that ϕ is a harmonic function. [3]

f) Write down the system of Navier-Stokes equations for an incompressible fluid flow with density ρ, viscocity ν, and velocity [3]

$$\mathbf{q} = u(x,y)\mathbf{i} + v(x,y)\mathbf{j}.$$

g) The complex potential of a two dimensional motion is given by

$$w = U\left(z^2 + 4z^{-2}\right),$$

where U is a real constant. Find expressions for the velocity potential $\phi(r,\theta)$ and stream functions $\psi(r,\theta)$. [4]

h) The velocity potential for a two dimensional flow is given by

$$\phi = \alpha \tan\left(\frac{y}{x}\right)$$

where α is a positive constant. Find the velocity field for the flow. [5]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

a) Show that the velocity field for a fluid motion given by

$$\mathbf{q} = (x + 2y + 4z)\mathbf{i} + (2x - 3y - z)\mathbf{j} + (4x - y + 2z)\mathbf{k}$$

represents a possible motion of an incompressible and irrotational fluid. [5]

b) Find the velocity potential for the flow in (a). [5]

c) Consider the two-dimensional velocity field given by

$$\mathbf{q} = \frac{y}{x^2 - 1}\mathbf{i} - \frac{x}{x^2 - 1}\mathbf{j}$$

Determine the equation of the streamline passing through the point $(4, 3)$ [5]

d) Find the velocity components for a flow with velocity potential given by

$$\phi = \frac{x}{x^2 + y^2}$$

[5]

QUESTION B3 [*20 Marks*]

Consider the boundary layer equations in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - au \qquad (2)$$

with boundary conditions

$$u = ax \quad , \quad v = -(\nu a)^{\frac{1}{2}} \quad \text{on y} = 0 \quad \text{and} \quad u = 0 \quad \text{on} \quad y = \infty$$

Using the similarity transformation $\eta = y\left(\frac{a}{\nu}\right)^{\frac{1}{2}}$ and the stream function formulation

$$\psi = x(\nu a)^{\frac{1}{2}} f(\eta)$$

where a is a constant and ν is the dynamic viscosity, Show that equation (2) and the boundary conditions can be transformed into

$$f''' + ff'' - (f')^2 - f' = 0$$

$$f = 1 \quad , \quad f' = 1 \quad \text{on } \eta = 0 \quad \text{and} \quad f' = 0 \quad \text{on } \eta = \infty$$

[20]

QUESTION B4 [*20 Marks*]

Consider the portion of fluid contained within any closed surface S fixed in space and containing a volume τ and the mass M of the fluid within the surface is

$$M = \int_{\tau} \rho d\tau.$$

Given that $\mathbf{q}$ is the fluid velocity at the element of dS, show that the equation of continuity is given by

$$\frac{d}{dt}(\log(\rho)) + \nabla \cdot \mathbf{q} = 0.$$

[20]

QUESTION B5 [*20 Marks*]

a) Find the complex velocity potential for the two dimensional irrotational incompressible flow

$$v = \frac{2y}{x^2+y^2}, \qquad u = \frac{2x}{x^2+y^2}.$$

[5]

b) i) Find the complex potential of a line sink of strength 2 at $z = 3 - 4i$. [2]

ii) Find the complex potential of a vortex street of strength 2π situated at

$$z = 4, 4 \pm \pi, 4 \pm 2\pi, 4 \pm 3\pi, \cdots,$$

[3]

c) The complex potential of a two dimensional motion is given by

$$w = k\left(z^n + a^{2n}z^{-n}\right),$$

where a and k are real constants. Find expressions for the velocity potential $\phi(r, \theta)$ and stream functions $\psi(r, \theta)$. [4]

d) Consider the motion specified by the velocity field

$$\mathbf{q} = \left[\left(\frac{a^2}{r^2} - 1\right)\cos\theta, \left(1 + \frac{a^2}{r^2}\right)\sin\theta, 0\right] \qquad \text{(cylindrical)}$$

Prove that the velocity field $\mathbf{q}$ represent possible motion of an incompressible fluid. [6]

QUESTION B6 [*20 Marks*]

a) Using Euler's equation of motion

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{q}^2\right) - \mathbf{q} \times (\nabla \times \mathbf{q}) = \mathbf{F} - \frac{1}{\rho}\nabla p,$$

show that Bernoulli's equation is given by

$$\frac{1}{2}\mathbf{q}^2 + \Lambda + \int \frac{dp}{\rho} - \frac{\partial \phi}{\partial t} = \beta(t),$$

where $\beta(t)$ is an arbitrary function, Λ and ϕ are scalar functions. Assume that the body forces are conservative and that the flow is of potential kind. [14]

b) Consider a homogenous, incompressible fluid such that the motion is steady. Derive the general Bernoulli equation for such a fluid. [6]

USEFUL FORMULAE

General Curvilinear Coordinates

If $f = f(u, v, w)$ and $\vec{r} = u\hat{e}_u + v\hat{e}_v + w\hat{e}_w$, then

$$h_u = \left|\frac{\partial \vec{r}}{\partial u}\right|, \qquad h_v = \left|\frac{\partial \vec{r}}{\partial v}\right|, \qquad h_w = \left|\frac{\partial \vec{r}}{\partial w}\right|,$$

and

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{e}_w.$$

If $\vec{q} = q_u\hat{e}_u + q_v\hat{e}_v + q_w\hat{e}_w$, then

$$\nabla \cdot \vec{q} = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}(q_u h_v h_w) + \frac{\partial}{\partial v}(h_u q_v h_w) + \frac{\partial}{\partial w}(h_u h_v q_w)\right),$$

and

$$\nabla \times \vec{q} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{e}_u & h_v\hat{e}_v & h_w\hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u q_u & h_v q_v & h_w q_w \end{vmatrix}$$

Cylindrical Coordinates

If $(u, v, w) = (r, \theta, z)$, then

$$h_r = 1, \qquad h_\theta = r, \qquad h_z = 1.$$

Spherical Coordinates

If $(u, v, w) = (\rho, \theta, \psi)$, then

$$h_\rho = 1, \qquad h_\theta = \rho, \qquad h_\psi = \rho\sin(\theta).$$

END OF EXAMINATION PAPER