UNIVERSITY OF SWAZILAND

EXAMINATION, 2016/2017

BASS II, B.Ed (Sec.) II, B.Sc. II

Title of Paper : Foundations of Mathematics

Course Number : MAT231/M231

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [40 *Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [40 *Marks*]

(a) Define each of the following.

i. A *proposition*. (2)

ii. A *tautology*. (2)

iii. A *relation* from a set A into a set B. (2)

iv. An *equivalence relation* on a set A. (4)

v. A *function* from a set A into a set B. (3)

vi. A *one-to-one* function $f : A \to B$. (3)

(b) Consider the statement:

"If it is raining today, then Sipho is wearing gumboots."

i. Write down (in English) the inverse of the statement. (2)

ii. Write down (in English) the converse of the statement. (2)

iii. Write down (in English) the contrapositive of the statement. (2)

(c) State the Generalized Principle of Mathematical Induction. (2)

(d) Write down the negation of each of the following statements.

i. $(\exists x \in \mathbb{R})(x^2 = 2)$. (2)

ii. $(\forall x \in \mathbb{Q})(\exists p \in \mathbb{Z})(\exists q \in \mathbb{Z})(x = p/q)$. (3)

(e) Show that $p \wedge \neg(p \to q) \equiv p \wedge \neg q$. (5)

(f) Consider the following predicates.

$$p(x) : x > -1$$
$$q(x) : x \in \{0, 1, 2\}.$$

Determine the truth values of the following propositions.

i. $p(-1) \to q(1)$. (2)

ii. $p(1) \wedge \neg p(-1)$. (2)

iii. $\neg(p(2) \vee q(2))$. (2)

END OF SECTION A – TURN OVER

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [20 *Marks*]

(a) Consider the predicate

$$p(x,y): \ x \neq y.$$

Determine the truth values of the following propositions.

i. $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})p(x,y)$. (2)

ii. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})p(x,y)$. (2)

iii. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})p(x,y)$. (2)

(b) Let $p(x,y)$ and $q(x,y)$ be predicates. Prove

$$\neg\left[(\forall x)(\forall y)(p(x,y) \rightarrow q(x,y))\right] \equiv (\exists x)(\exists y)(p(x,y) \wedge \neg q(x,y))$$

(5)

(c) Determine whether the following argument is valid or invalid.

$$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow p \\ \therefore & p \vee q. \end{array}$$

(5)

(d) Prove

$$p \rightarrow q \equiv \neg q \rightarrow \neg p.$$

(4)

QUESTION B3 [20 *Marks*]

(a) Prove: *For every integer x, x^2 is even if and only if x is even.* (7)

(b) Prove: *The number $\sqrt{2}$ is irrational.* (7)

(c) Let $a \neq 0, b \neq 0$ and c be integers. Prove:

i. *If $a \mid b$ and $a \mid c$, then $a \mid (b+c)$.* (3)

ii. *If $a \mid b$ and $b \mid c$, then $a \mid c$.* (3)

QUESTION B4 [*20 Marks*]

(a) Let A and B be sets in a universal set U. Prove each of the following.

i. If $A \subseteq B$, then $A \cup B = B$ (4)

ii. $(A \cap B)^c = A^c \cup B^c$. (6)

(b) i. Define a *partition* of a set A. (2)

ii. Let $A = \{1,2,3,4,5,6\}$, $A_1 = \{1\}, A_2 = \{2,3\}, A_3 = \{4,5,6\}$. Show that $\{A_1, A_2, A_3\}$ is a partition of A. (3)

(c) Let $B = \{1,2\}$ and $C = \{3,4\}$. Find

i. $\mathscr{P}(B)$. ii. $\mathscr{P}(B \cap C)$. (2,3)

QUESTION B5 [*20 Marks*]

(a) Let $X = \mathbb{Z}^+$ be the set of non-negative integers and define a relation R on X by mRn if and only if $m \mid n$. Prove that R is antisymmetric. (5)

(b) Define a relation $\sim$ on $\mathbb{Z}$ by $m \sim n$ if and only if $m \equiv n \pmod 2$.

i. Show that $\sim$ is an equivalence relation on $\mathbb{Z}$. (7)

ii. List the equivalence classes of $\mathbb{Z}$ given by $\sim$. (2)

(c) Let $\mathscr{A}$ be a collection of sets. Let R be the relation on $\mathscr{A}$ defined by $(A, B) \in R$ if and only if $A \subseteq B$. Show that $\mathscr{A}$ with this relation is a poset. (6)

QUESTION B6 [*20 Marks*]

(a) Let $f(n) = 3^{2n} + 7$. Use mathematical induction to prove that $f(n)$ is divisible by 8 for all integers $n \geq 0$. (6)

(b) Use strong induction to prove: *Any integer $n > 1$ can be written as a product of prime numbers.* (6)

(b) i. Prove that the composition of two injective functions is also injective. (4)

ii. Prove that the composition of two surjective functions is also surjective. (4)

END OF EXAMINATION PAPER