University of Swaziland

Main Examination, 2017/2018

BASS IV, B.Ed (Sec.) IV, B.Sc IV

Title of Paper : NUMERICAL ANALYSIS II

Course Number : M411

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) i. Define a Lipschitz constant. [3 marks]

ii. Consider the function $f(t,y) = \sqrt{y}$ on the interval $y \in [0,\infty)$. Prove that the function does not satisfy the Lipschitz condition. [3 marks]

(b) Use Heun's method with step size $h = 1$ to solve the initial value problem

$$\frac{dy}{dt} = \frac{1}{2}(y+t) \qquad \text{on } [0,3]$$

with initial condition $y(0) = 1$ [7 marks]

(c) Consider the IVP

$$\begin{aligned} x' &= 2 - 4x - 2y, \qquad x(0) = 1, \\ y' &= 1 + x - y, \qquad y(0) = 0, \end{aligned}$$

Use the Euler's method with $h = 0.1$ to find the solution of $x(0.2)$ and $y(0.2)$. [5 marks]

(d) Consider the following scheme

$$y_{i+1} - y_{i-1} = \frac{h}{3}[7f(t_i,y_i) - 2f(t_{i-1},y_{i-1}) + f(t_{i-2},y_{i-2})]$$

for solving the initial value problem

$$y' = f(t,y)$$

with h denoting the time-step $h = t_{i-1} - t_i$. Show that this scheme is third order accurate. [6 marks]

(e) Find the linear least squares function, that best fits the curve $y = \sqrt{2x+1}$ over the interval $0 \leq x \leq \frac{3}{2}$. [8 marks]

(f) Use the Von Neumann analysis to obtain a restriction on the time step size that guarantees stability of the finite difference solution, based on forward difference in time and central difference in space, for solving the heat equation $u_t = u_{xx}$ [8 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Consider the initial value problem

$$y' = t|y|, \quad y(1) = 0$$

Discuss the existence and uniqueness of the solution of the IVP on the interval $0 \leq t \leq 2$ [6 marks]

(b) Without solving, derive the Taylor method of order 3 that corresponds to the IVP [6 marks]

$$y' = y\cos(t), \quad y(0) = 1$$

(c) Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

and the following Runge-Kutta method for solving the differential equation

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(K_1 + 4K_2 + K_3) \\ K_1 &= hf(x_n, y_n) \\ K_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right) \\ K_3 &= hf\left(x_n + h, y_n - K_1 + 2K_2\right) \end{aligned}$$

Compute $y(0.4)$ when

$$\frac{dy}{dx} = \frac{y + x}{y - x}, \qquad y(0) = 1$$

and $h = 0.2$ [8 marks]

QUESTION B3 [*20 Marks*]

B3 Consider the differential equation

$$\frac{\partial u}{\partial t} = \beta^2 \frac{\partial^2 u}{\partial x^2} - \alpha u$$

where α and β are known constants.

(a) Discretize the equation using the forward difference scheme in time and the central difference scheme in space. [5 marks]

(b) Show that the scheme derived in a) is consistent and give the order of error in space and time. [15 marks]

QUESTION B4 [*20 Marks*]

B4 A function U satisfies the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - 32U = 0$$

at every point inside the square bounded by the straight lines $x = \pm 1$, $y = \pm 1$ and is subject to the boundary conditions :

$$\begin{array}{lll} U = 0 & \text{on} \;\; y = 1, & -1 \le x \le 1 \\ U = 1 & \text{on} \;\; y = -1, & -1 \le x \le 1 \\ \partial U/\partial x = -\frac{1}{2}U & \text{on} \;\; x = 1, & -1 \le y \le 1 \\ \partial U/\partial x = \frac{1}{2}U & \text{on} \;\; x = -1, & -1 \le y \le 1 \end{array}$$

Derive a finite-difference solution scheme for the above problem using step sizes $\Delta x = \Delta y = 1$, and central difference formulas for the derivatives and present the scheme in the form $AU = K$ [DO NOT SOLVE!!!]. [20 marks]

QUESTION B5 [*20 Marks*]

B5 (a) Find an equation of the form $f(x) = a\sqrt{x} + bx^2$ that best approximates the data in the table

x	0	1	4
y	1	4	8

in the least squares sense. [10 Marks]

(b) Find the continuous least squares trigonometric polynomial $S_2(x)$ for $f(x) = 1 - x$ on $[-\pi, \pi]$ [10 Marks]

QUESTION B6 [*20 Marks*]

B6 Derive the formula for the Adams-Bashforth Three-Step Explicit Method and prove that the method is convergent. [20 marks]

END OF EXAMINATION